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CONVEXITY OF DOMAIN FUNCTIONALS

BY

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CONVEXITY OF DOMAIN FUNCTIONALS

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P. R. Garabedian and M. Schiffer

CHAPTER I

INTRODUCTION

1. Outline of the paper.

A problem of considerable significance is the study of the dependence of the Green's function, Neumann's function, and eigenfunctions of a linear elliptic partial differential equation on their domains of definition. The importance of this question lies in the difficulty which is generally encountered in the explicit calculation of such functions. Since elementary formulas are the exception rather than the rule in the theory of partial differential equations in an arbitrary domain, one turns to the investigation of the properties of the basic solutions, and variational formulas exhibiting the domain dependence of these solutions furnish one of the principal tools of the investigation.

One of the most common applications of the theory of variation of the Green's function, Neumann's function and eigenfunctions arises in the study of extremal problems for the capacity, virtual mass and eigenvalues of a domain. We are led to such extremal problems on the one hand in an attempt to estimate these domain functionals in terms of the geometry of the domain, and on the other hand by the equivalence of the solution of particular variational problems with the existence and uniqueness of solutions of physical problems, such as the construction of free boundary flows. Extensive investigations in

these two directions have been carried through for Laplace's equation in the plane and also for other typical elliptic partial differential equations in two independent variables [2, 7, 8, 9, 18].

In this paper, we develop a rigorous theory of variation of domain functions in space of three dimensions as well as in the plane. We not only present an adequate mathematical discussion of the classical Hadamard variational formulas in space, but also generalize the so-called interior variational method to three dimensions. We derive expressions for the second variations of the capacity, virtual mass, and other physical quantities already mentioned, and we deduce from them various interesting convexity theorems for these domain functionals.

Our investigation of the second variation was motivated by the suggestion of Max Shiffman that in cases where one can guess a domain for which a certain combination of domain functionals is stationary, one might be able to apply a minimax theory in order to prove that the domain in question actually minimizes that combination, provided one could show that whenever the first variation of the combination vanishes the second variation is positive-definite. This suggestion is based on the fact that on certain surfaces, corresponding to two points whose heights are relative minima there exists a saddle-point. While we have not had any direct success with this line of reasoning, we have been able to deduce a number of uniqueness theorems from convexity properties of the domain functionals which are based on the second variation. One should mention in this connection the work of Friedrichs [6], who proved by such a method the uniqueness of certain free boundary flows.

In Chapter II we define interior variations of a 3 -dimensional domain D by means of differentiable mappings of D depending on a small parameter ε.

The first order shifts in terms of ε of the Green's function, Neumann's function and eigenvalues which result from this variation of D are calculated rigorously by referring all varied quantities back to the original domain D through the infinitesimal mappings. The study of the domain dependence of these basic functions for a given linear elliptic partial differential equation is thus transformed into an investigation in the fixed domain D of the dependence of the solution on the coefficients of the equation. Such an investigation is readily carried through by Hilbert's classical method based on integral equations [11]. The variational formulas which result, for example, in the case of the capacity, are given in terms of domain integrals involving the Maxwell tensor of the relevant electrostatic field, and their validity does not depend upon strong smoothness assumptions on the boundary of the domain. However, when the boundary of the domain is sufficiently smooth, our interior variational formulas yield by application of the divergence theorem the classical Hadamard variational formulas, for which we thus obtain a strict derivation. Our 3 -dimensional variational theory collapses easily by specialization to the better known theory of variation in the plane.

Once in possession of a rigorous proof by the Hilbert method that we can expand the varied domain functions in powers of ε, we are justified in employing the perturbation method to calculate the second variations of these functions. We do this in Chapter III and obtain interesting second variation expressions for the capacity, virtual mass and eigenvalues corresponding to various particular ways in which we can shift D. A number of convexity theorems for these domain functionals are the outgrowth of this investigation. For example, if we shift the surface of D along level surfaces of a harmonic function U, the capacity of D with respect to a fixed interior point turns out

to be a convex function of U. Similarly, the capacity of a convex domain D is a convex function of a parameter which gives a variation of D defined in terms of the support function of the surface of D. These convexity theorems are applied to establish extremal properties of domains for which it is known that the capacity is stationary under certain constrained variations, and thus a number of uniqueness theorems are deduced. Such a theorem can be obtained also for the problem due to Evans [5] of finding a surface of least capacity enclosing a given curve.

In Chapter IV we specialize our variational theory to the case of two independent variables in order to apply it to show the existence of vortex sheets in axially symmetric, irrotational flow of an incompressible fluid. An indication of these results will be given in the next section, where we sketch heuristically an extremal characterization of vortex sheets in 3 -dimensional space without symmetry of any kind. In the axially symmetric case, the convexity of the virtual mass in dependence on the domain can also be used to discuss the extremal characterization of a vortex sheet.

Chapter V is devoted to the study of the eigenfunctions and eigenvalues of the vibrating membrane. Using the second variation, we show that under certain conformal mappings of a domain depending on a suitable parameter, the inverse square of the principal frequency of the domain becomes a convex function of the parameter. This theorem is applied to show that for fixed principal frequency of a domain, the capacity with respect to an interior point is a maximum when the domain is a circle about the point, a theorem due to Polya and Szegö [16].

2. Vortex sheets.

In order to motivate the detailed study of variational methods in 3-dimensional space which follows in later chapters, we treat heuristically in this section a 3 -dimensional extremal problem for the virtual mass of a steady irrotational flow of an incompressible fluid whose solution yields a flow past a vortex sheet. We have only succeeded in proving the existence and uniqueness of vortex sheets from this extremal characterization in the cases of plane and axially symmetric flow, but a formal discussion of the general 3 -dimensional case, which is the one of interest in aerodynamic theory, should indicate the direction in which further development of the material in this paper should be pushed.

Let C be a simple closed curve in space and let Σ be a closed surface which intersects every curve looped around C and bounds an infinite domain D. Let $\varphi = \varphi(x,y,z)$ be a harmonic function in D with the expansion

$$\varphi = x + \frac{\alpha x}{r^3} + \dots , \qquad r^2 = x^2 + y^2 + z^2 \qquad , \tag{1.2.1}$$

near infinity and with an inner normal derivative which vanishes on Σ,

$$\frac{\partial \varphi}{\partial \nu} = 0 \quad . \tag{1.2.2}$$

The function φ represents the velocity potential of a steady 3 -dimensional flow past Σ of an incompressible fluid in D. The coefficient α in (2.1) is related to the kinetic energy of the flow and we shall call it the virtual mass of Σ with respect to the x direction.

We show formally that if, for a fixed curve C, the surface Σ is so chosen that

$$\alpha = \text{minimum} \qquad , \tag{1.2.3}$$

then Σ reduces to a vortex sheet enclosed by the curve C.

Let Σ^* be another surface spanned through C whose normal displacement $\delta\nu$ from Σ, measured from any point of Σ, is infinitesimally small. We denote by D^* the infinite domain bounded by Σ^*, we denote by φ^* the velocity potential of the flow of the form (2.1) past Σ^*, and we denote by α^* the virtual mass of this flow. We attempt to estimate the difference $\alpha^* - \alpha$ in terms of the normal shift $\delta\nu$.

If R denotes the surface of a large sphere enclosing Σ and Σ^*, we find by Green's theorem that

(1.2.4) $$\alpha^* - \alpha = -\frac{1}{4\pi}\iint_R \left\{\varphi\frac{\partial\varphi^*}{\partial\nu} - \varphi^*\frac{\partial\varphi}{\partial\nu}\right\} d\sigma \quad ,$$

since the integral is independent of the radius of the sphere R. Here $d\sigma$ denotes the surface element. A further application of Green's theorem yields

(1.2.5) $$\alpha^* - \alpha = \frac{1}{4\pi}\iiint_{D-D^*} (\nabla\varphi)^2 d\tau + \frac{1}{4\pi}\iiint_{D^*} [\nabla\varphi^* - \nabla\varphi]^2\, d\tau \quad ,$$

where $d\tau$ is the volume element. From (2.5), it is clear that the first order term $\delta\alpha$ of the difference $\alpha^* - \alpha$, considered as a functional depending on $\delta\nu$, is

(1.2.6) $$\delta\alpha = \frac{1}{4\pi}\iint_\Sigma (\nabla\varphi)^2\,\delta\nu\, d\sigma \quad .$$

Let us now suppose that Σ is an extremal surface for the minimum problem (2.3). We first conclude that Σ must reduce to a single sheet containing no interior points, since according to (2.5) the virtual mass α decreases monotonically as the flow region D expands. Secondly, we notice that $(\nabla\varphi)^2$ must have identical values on both sides of the extremal sheet Σ, since the first variation (2.6) must vanish for every normal shift of the extremal surface, and since a normal shift of the sheet Σ corresponds to values of $\delta\nu$ which

differ in sign only on the opposite sides of Σ. Thus in the extremal case, $(\nabla\varphi)^2$ is continuous across Σ, which implies by Bernoulli's law,

$$\frac{1}{2}(\nabla\varphi)^2 + p = \text{const.} \quad , \tag{1.2.7}$$

that the pressure distribution p is continuous through Σ. But the continuity of the pressure p is precisely the physical condition which characterizes Σ as a vortex sheet. In space, the velocities on the two sides of a vortex sheet are permitted to have, within the tangent plane of the sheet, quite unrelated directions, but their magnitudes must be equal. In 3 -dimensional aerodynamic theory, such sheets are introduced to account for the lift and drag produced on an airfoil. We have succeeded here in connecting this concept with the minimum energy principle (2.3).

CHAPTER II

THE METHOD OF INTERIOR VARIATION

1. Generalities.

We want to develop in this chapter the theory of the variation for Green's functions with respect to linear partial differential equations of elliptic type under a variation of their domain of definition. For the sake of simplicity we shall treat the case of three independent variables x_i $(i = 1,2,3)$ which vary in a domain D_o of three-dimensional space. Let $p(x_i)$ be a continuously differentiable function in this basic domain D_o and consider the partial differential equation of elliptic type

$$(2.1.1) \qquad L[u] = \nabla^2 u - pu = 0 \quad , \quad \nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \quad .$$

Let D be a subdomain of D_o and let us assume, at first, that D has a smooth boundary C. We assume further that the only solution of (1.1) in D which vanishes on C is the trivial solution $u \equiv 0$. In this case, there exists a Green's function of the domain D with respect to (1.1). This Green's function $G(P,Q)$ (with $P \equiv x_i$, $Q \equiv \xi_i$) is characterized by the following three requirements:

a) For fixed $Q \in D$, $G(P,Q)$ is a solution of (1.1) as a function of P and twice continuously differentiable in D, except at the point Q.

b) The function $G(P,Q) - \frac{1}{4\pi\, r(P,Q)}$ is continuous in D, has uniformly bounded first derivatives with respect to P, except at $P = Q$, and its second derivatives grow at most like $r(P,Q)^{-1}$ if P approaches Q, where

$$(2.1.2) \qquad r(P,Q)^2 = \sum_{i=1}^{3} (x_i - \xi_i)^2$$

c) $G(P,Q)$ vanishes for $P \in C$, $Q \in D$.

It is well known that these requirements determine the Green's function in a unique way and that the Green's function satisfies the symmetry condition

(2.1.3) $$G(P,Q) = G(Q,P) \quad .$$

Thus, in particular, Green's function is a solution of (1.1) in dependence on the parameter point Q also.

If u(P) is an arbitrary solution of (1.1) in D and is continuous in D+C, it can be expressed in terms of its boundary values on C by means of the Green's function in the form

(2.1.4) $$u(P) = \iint_C \frac{\partial G(P,Q)}{\partial \nu_Q} u(Q) d\sigma_Q \quad ,$$

where ν_Q denotes the interior normal at the point Q with respect to the surface C and where $d\sigma_Q$ is the surface element at Q.

We can also solve the inhomogeneous differential equation

(2.1.5) $$\nabla^2 u - pu = f(x_i) \quad , \qquad u = 0 \text{ on } C \quad ,$$

if $f(x_i)$ is Hölder continuous in D+C. In fact, the solution u(P) can be represented in the form

(2.1.6) $$u(P) = -\iiint_D G(P,Q) f(Q) d\tau_Q \quad ,$$

where $d\tau_Q$ is the volume element at the point Q.

This result leads to an interesting interpretation of the quadratic form

(2.1.7) $$\Gamma[f,f] = \iiint_D \iiint_D G(P,Q) f(P) f(Q) d\tau_P d\tau_Q \quad .$$

In fact, we obtain from (2.1.5) and (2.1.6) by means of Green's identity

(2.1.8) $$\Gamma[f,f] = \iiint_D [(\nabla u)^2 + pu^2] d\tau \quad .$$

The right-hand side is just the Dirichlet integral of u with respect to the differential equation (1.1). In the special case that $p(x_i)$ is non-negative

in D, the Green's function $G(P,Q)$ is the kernel of a positive-definite form. This fact makes the theory of (1.1) particularly simple for the case of a non-negative coefficient $p(P)$.

2. Interior variations.

We consider a three times continuously differentiable vector field in D_o.

$$(2.2.1) \qquad S_i = S_i(x_j) \qquad i,j = 1,2,3,$$

and the transformation

$$(2.2.2) \qquad x_i^* = x_i^*(x_j;\varepsilon) = x_i + \varepsilon S_i(x_j) \qquad ,$$

which depends on the real parameter ε. Given a proper subdomain $D \subset D_o$, we may choose a bound $b(D,D_o) > 0$ such that for $|\varepsilon| < b(D,D_o)$ the domain D is mapped topologically onto another proper subdomain $D^* \subset D_o$ with smooth boundary surface C^*. At the same time, we can choose $b(D,D_o)$ so small that all domains D^* obtained still possess a Green's function $G^*(P,Q)$ with respect to the differential equation (1.1). Our principal aim in this chapter is to express $G^*(P,Q)$ in terms of the original Green's function $G(P,Q)$ and the transformation vector field (2.1).

For this purpose, we refer the function $G^*(P,Q)$ back to the original domain D. Let $P^*(P) \equiv x_i^*(x_j)$, $Q^*(Q) \equiv \xi_i^*(\xi_j)$ and consider the function

$$(2.2.3) \qquad g(P,Q;\varepsilon) = G^*[P^*(P),Q^*(Q)] \qquad ,$$

which is well-defined in D, twice continuously differentiable except for $P = Q$, and which vanishes if either argument point lies in the boundary surface C of D.

Let $u^*(P)$ be an arbitrary solution of (1.1) in D^* and let

$$(2.2.4) \qquad u(P;\varepsilon) = u^*(P^*(P))$$

be the corresponding function in D. We have

$$(2.2.5) \qquad \frac{\partial u^*}{\partial x_j^*} = \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}\,\frac{\partial x_i}{\partial x_j^*} \qquad , \qquad j = 1,2,3 \qquad .$$

The function $u^*(P^*)$ yields a stationary value to the Dirichlet integral

$$(2.2.6)\quad \iiint_{D^*} [(\nabla u^*)^2 + pu^{*2}]d\tau^* = \iiint_D \left\{ \sum_{j=1}^{3} \left(\sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial x_j^*} \right)^2 + pu^2 \right\} \theta \, d\tau$$

for all functions in D^* with same boundary values on C^*. Here

$$(2.2.7)\qquad \theta(x_i) = \frac{\partial(x_1^*, x_2^*, x_3^*)}{\partial(x_1, x_2, x_3)}$$

is the Jacobian of the transformation (2.2). Consequently, the function $u(P;\mathcal{E})$ must satisfy the Euler-Lagrange equation for the right-hand integral in (2.6). We introduce the notation

$$(2.2.8)\qquad A_{ik} = \theta(x_\ell) \sum_{j=1}^{3} \frac{\partial x_i}{\partial x_j^*} \frac{\partial x_k}{\partial x_j^*} = A_{ki} \qquad ,$$

$$(2.2.8')\qquad \rho(x_i) = \theta(x_i) p(x_i^*) \qquad ,$$

and we obtain the following transformed differential equation for $u(P;\mathcal{E})$:

$$(2.2.9)\qquad L_\varepsilon[u] = \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i} [A_{ik} \frac{\partial u}{\partial x_k}] - \rho u = 0 \qquad .$$

This equation is satisfied, in particular, by $g(P,Q;\varepsilon)$ in dependence on P for $P \in D$, $P \neq Q$.

The differential equation (2.2.9) was obtained as the Euler-Lagrange equation with respect to the Dirichlet integral

$$(2.2.10)\qquad Q_\varepsilon[u] = \iiint_D [\sum_{i,k=1}^{3} A_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} + \rho u^2] d\tau \qquad .$$

This integral reduces obviously to

$$(2.2.10')\qquad Q_o[u] = \iiint_D [(\nabla u)^2 + pu^2] d\tau$$

in the case $\varepsilon = 0$; $Q_\varepsilon[u]$ will play an important role in the subsequent considerations.

The preceding transformation of (1.1) into (2.9) permits us to consider the Green's functions of varying domains with respect to the fixed differential equation (1.1) also as the Green's functions of the fixed domain D with respect to the varying differential equation (2.9). In this way, the dependence of the Green's function on the parameter ε can be investigated in a more convenient way; in particular, the general theory of linear integral equations can now be brought into play in an easier fashion.

3. The parametrix.

We have to study the character of the singularity of $g(P,Q;\varepsilon)$ if P moves to Q. We refer the distance function $r(P^*,Q^*)$ back to the domain D and find

$$r(P^*(P),Q^*(Q)) = \Big(\sum_{j=1}^{3} (x_j^* - \xi_j^*)^2\Big)^{1/2} \tag{2.3.1}$$

$$= \Big[\sum_{i,k=1}^{3} a_{ik}(\xi_j)(x_i - \xi_i)(x_k - \xi_k) + \frac{1}{2}\sum_{i,k,\ell=1}^{3} \frac{\partial a_{ik}(\xi_j)}{\partial \xi_\ell}(x_i-\xi_i)(x_k-\xi_k)(x_\ell-\xi_\ell)$$

$$+ O(r^4)\Big]^{1/2} \quad , \qquad r = r(P,Q) \quad ,$$

with

$$a_{ik}(x_\ell) = \sum_{j=1}^{3} \frac{\partial x_j^*}{\partial x_i}\,\frac{\partial x_j^*}{\partial x_k} \quad . \tag{2.3.2}$$

By virtue of the continuous differentiability of $G(P^*,Q^*) - \frac{1}{4\pi} r(P^*,Q^*)^{-1}$ and the character of the transformation (2.2), we can assert that

$$g(P,Q;\varepsilon) = \frac{1}{4\pi\, r(P^*(P),Q^*(Q))} + h(P,Q;\varepsilon) \quad , \tag{2.3.3}$$

where $h(P,Q;\varepsilon)$ has continuous first derivatives in D.

The function $\frac{1}{4\pi} r(P^*,Q^*)^{-1}$ has for P near Q the same asymptotic behavior as $g(P,Q;\varepsilon)$ in the sense that the difference function $h(P,Q;\varepsilon)$ is continuously differentiable in D. It can even be shown that the second derivatives of $h(P,Q;\varepsilon)$ become infinite at most like $r(P,Q)^{-1}$ if P approaches Q. We can

construct another function of P and Q which is somewhat simpler and has the same asymptotic behavior. In fact, the function

$$(2.3.4)\qquad \delta_{\varepsilon}(P,Q) = \frac{1}{4\pi}\Big[\frac{1}{2}\sum_{i,k=1}^{3}(a_{ik}(\xi_j) + a_{ik}(x_j))(x_i - \xi_i)(x_k - \xi_k)\Big]^{-1/2}$$

satisfies our requirements, as is immediately seen by series development near the point Q.

We define now the parametrix functions $s(P,Q;\varepsilon)$ with respect to the differential equation (2.2.9) as follows:

a) The parametrix $s(P,Q;\varepsilon)$ is a symmetric function of both arguments.

b) We can write

$$(2.3.5)\qquad s(P,Q;\varepsilon) = \delta_{\varepsilon}(P,Q) + R_{\varepsilon}(P,Q) \quad ,$$

where $R_{\varepsilon}(P,Q)$ is twice continuously differentiable in D for $P \neq Q$, has uniformly bounded first derivatives for $P \neq Q$, and has second derivatives which become infinite at most like $r(P,Q)^{-1}$ for $P \to Q$.

c) The parametrix vanishes if, for fixed $Q \in D$, the argument point P lies on the boundary surface C.

The construction of such a parametrix can be performed in various ways. The concept of the parametrix was introduced by Hilbert [11], who applied it to study the dependence of solutions of partial differential equations on parameters which occur in the coefficients of the equation. While it may be difficult to obtain a fundamental solution for a given differential equation, the construction of a parametrix requires no comparable efforts, since only the boundary condition and the singular character have to be observed.

4. The integral equations for $g(P,Q;\varepsilon)$.

Let $Q[u,v]$ be the bilinear form belonging to the quadratic functional (2.2.10). Let u and v be two functions which are twice continuously

differentiable in D+C. We have by Green's first identity

(2.4.1) $$Q_{\varepsilon}[u,v] = -\iint_C vP_{\varepsilon}(\nu,\nabla u)d\sigma - \iiint_D vL_{\varepsilon}[u]d\tau \quad ,$$

where

(2.4.2) $$P_{\varepsilon}(\nu,\nabla u) = \sum_{i,k=1}^{3} A_{ik}\nu_i \frac{\partial u}{\partial x_k} \quad ,$$

and ν is the inner normal to C with components ν_i. From this result follows immediately Green's second identity

(2.4.3) $$\iiint_D (vL_{\varepsilon}[u] - uL_{\varepsilon}[v])d\tau = -\iint_C (vP_{\varepsilon}(\nu,\nabla u) - uP_{\varepsilon}(\nu,\nabla v))d\sigma \quad .$$

We apply, at first, (2.4.1) to the two functions

(2.4.4) $$u = g(P,Q;\varepsilon) \quad , \quad v = g(P,R;\varepsilon_0) \quad , \quad Q,R \in D \quad .$$

We have to modify the identity in the usual way by eliminating from the domain of integration small spheres around the points Q and R where the integrand is singular. If we let the radii of the two safety spheres tend to zero, the surface integral over the sphere around R will disappear, since v tends to infinity only like the reciprocal of its radius, while its surface area tends to zero like the square of its radius. The contribution of the sphere around Q is given by

(2.4.5) $$g(Q,R;\varepsilon_0)\ \frac{1}{4\pi}\iint_U \frac{\sum_{i,k,\ell=1}^{3} A_{ik}(\xi_j)a_{k\ell}(\xi_j)\,\nu_i\,\nu_\ell}{\left(\sum_{i,k=1}^{3} a_{ik}(\xi_j)\,\nu_i\,\nu_k\right)^{3/2}}\,dw \quad ,$$

where dw is the surface element of the unit sphere U around the point Q. From definitions (2.8) and (3.2) follows

(2.4.6) $$\sum_{k=1}^{3} A_{ik}(\xi_j)a_{k\ell}(\xi_j) = \delta_{i\ell}a(\xi_j) \quad .$$

There is no restriction of generality if we assume

$$(2.4.7)\qquad \sum_{i,k=1}^{3} a_{ik}(\xi_j)\,\nu_i\nu_k = \sum_{i=1}^{3} A_i\,\nu_i^2 \quad ,$$

that is, that the quadratic form is at Q on principal axes. It is well known from the theory of the attraction of ellipsoids that

$$(2.4.8)\qquad J(A_\nu) = \iint_U \frac{d\omega}{[A_1\nu_1^2 + A_2\nu_2^2 + A_3\nu_3^2]^{1/2}} = 2\pi\int_0^\infty \frac{dt}{\sqrt{(A_1+t)(A_2+t)(A_3+t)}} \quad .$$

Hence, we obtain easily

$$(2.4.9)\qquad \sum_{\nu=1}^{3} \frac{\partial J}{\partial A_\nu} = -\frac{1}{2}\iint_U \frac{d\omega}{[A_1\nu_1^2 + A_2\nu_2^2 + A_3\nu_3^2]^{3/2}} = -2\pi\cdot(A_1A_2A_3)^{-1/2} \quad .$$

Thus, we have

$$(2.4.10)\qquad \frac{1}{4\pi}\iint_U \frac{d\omega}{[\sum_{i,k=1}^{3} a_{ik}(\xi_j)\,\nu_i\nu_k]^{3/2}} = \theta(\xi_j)^{-1} \quad ,$$

and the contribution of the sphere around Q becomes simply $g(Q,R;\varepsilon_o)$.

We have further $L_\varepsilon[u] \equiv 0$ in D, whence we derive finally

$$(2.4.11)\qquad g(Q,R;\varepsilon_o) = Q_\varepsilon[g(P,Q;\varepsilon),g(P,R;\varepsilon_o)] \quad .$$

In exactly the same way, we find

$$(2.4.11')\qquad g(Q,R;\varepsilon) = Q_{\varepsilon_o}[g(P,Q;\varepsilon),g(P,R;\varepsilon_o)] \quad .$$

We define now the difference terms

$$(2.4.12)\qquad A_{ik}^{(\varepsilon,\varepsilon_o)} = A_{ik}(x_\nu;\varepsilon) - A_{ik}(x_\nu;\varepsilon_o) \quad ,$$

$$\rho^{(\varepsilon,\varepsilon_o)} = \rho(x_\nu;\varepsilon) - \rho(x_\nu;\varepsilon_o)$$

and the corresponding bilinear form

$$(2.4.13)\qquad E^{(\varepsilon,\varepsilon_o)}[u,v] = \sum_{i,k=1}^{3} A_{ik}^{(\varepsilon,\varepsilon_o)} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_k} + \rho^{(\varepsilon,\varepsilon_o)}\,uv \quad .$$

Subtracting (4.11) from (4.11'), we then obtain

$$g(Q,R;\varepsilon)-g(Q,R;\varepsilon_0) = -\iiint_D E^{(\varepsilon,\varepsilon_0)}[g(P,Q;\varepsilon),g(P,R;\varepsilon_0)]d\tau \quad . \tag{2.4.14}$$

This is an integro-differential equation for the Green's function $g(Q,R;\varepsilon)$ in terms of the Green's function $g(Q,R;\varepsilon_0)$. It is very simple and symmetric, but has the disadvantage that the coefficients of the integro-differential equation are highly singular and thus this equation does not exhibit clearly the continuous character of g in dependence on ε.

In order to overcome this difficulty, we apply now (4.3) with

$$u = g(P,Q;\varepsilon) \quad , \quad v = s(P,R;\varepsilon) \quad , \quad Q,R\in D \quad , \tag{2.4.15}$$

where s is any admissible parametrix. We have to exclude again the singular points Q and R by safety spheres and to pass then to the limit of vanishing radius. As before, we find easily that the contribution of the spheres around Q and R are $s(Q,R;\varepsilon)$ and $-g(R,Q;\varepsilon)$, respectively. Using the fact that g is a solution of (2.9) and is symmetric in its arguments, we thus obtain

$$g(Q,R;\varepsilon)-s(Q,R;\varepsilon) = \iiint_D g(P,Q;\varepsilon)L_\varepsilon[s(P,R;\varepsilon)]d\tau_P \quad . \tag{2.4.16}$$

This equation is an integral equation of the second kind for $g(Q,R;\varepsilon)$ with an admissible kernel for the Fredholm theory. In fact, we may put

$$L_\varepsilon[s(P,R;\varepsilon)] = L_\varepsilon[s(P,R;\varepsilon)-g(P,R;\varepsilon)] \quad , \tag{2.4.17}$$

and by definition of the parametrix the second derivatives of s-g become infinite of at most the same order as $r(P,R)^{-1}$ if $P\to R$. However, we want to transform the integral equation into such a form that it may be resolved by a Neumann's series.

For this purpose, we apply (4.3) again with

$$u = g(P,Q;\varepsilon) \quad , \quad v = g(P,R;\varepsilon_0)-s(P,R;\varepsilon_0) \quad . \tag{2.4.18}$$

Observe that v has now bounded first derivatives in D, even near the point R. Hence, there is no contribution from the safety sphere around R and we obtain, since $L_\varepsilon[u] = 0$,

$$(2.4.19)\quad g(Q,R;\varepsilon_0) - s(Q,R;\varepsilon_0) = -\iiint_D L_\varepsilon[g(P,R;\varepsilon_0) - s(P,R;\varepsilon_0)]g(P,Q;\varepsilon)d\tau_P .$$

Subtracting (4.19) from (4.16), we obtain

$$(2.4.20)\quad g(Q,R;\varepsilon) - g(Q,R;\varepsilon_0) = s(Q,R;\varepsilon) - s(Q,R;\varepsilon_0) + \iiint_D K^{(\varepsilon,\varepsilon_0)}(P,R)g(P,Q;\varepsilon)d\tau_P ,$$

with

$$(2.4.21)\quad K^{(\varepsilon,\varepsilon_0)}(P,R) = L_\varepsilon[g(P,R;\varepsilon_0)] - L_{\varepsilon_0}[g(P,R;\varepsilon_0)] + L_\varepsilon[s(P,R;\varepsilon) - s(P,R;\varepsilon_0)] .$$

Up to this point, no particular assumption was made concerning the dependence of the parametrix upon the parameter ε. We observe that the $a_{ik}(x_j;\varepsilon)$ depend analytically upon ε and that we can assume without loss of generality that the parametrix depends on ε in a sufficiently smooth manner. Under this assumption, the kernel $K^{(\varepsilon,\varepsilon_0)}$ is small of the order $\varepsilon - \varepsilon_0$, except near the point R, where it becomes infinite. But we may write

$$(2.4.22)\quad K^{(\varepsilon,\varepsilon_0)}(P,R) = L_\varepsilon[g(P,R;\varepsilon_0) - s(P,R;\varepsilon_0)] + L_\varepsilon[s(P,R;\varepsilon) - g(P,R;\varepsilon)] ,$$

and in view of the characteristic property of the parametrix we may conclude that $K^{(\varepsilon,\varepsilon_0)}(P,R)$ becomes infinite at most like $r(P,R)^{-1}$ if $P \to R$. For $|\varepsilon - \varepsilon_0|$ small enough, the Neumann's series for the reciprocal kernel of $K^{(\varepsilon,\varepsilon_0)}$ will converge.

Let us put

$$(2.4.23)\quad \gamma(Q,R;\varepsilon,\varepsilon_0) = g(Q,R;\varepsilon_0) + s(Q,R;\varepsilon) - s(Q,R;\varepsilon_0) .$$

We have

$$(2.4.24)\quad K^{(\varepsilon,\varepsilon_0)}(Q,R) = L_\varepsilon[\gamma(Q,R;\varepsilon,\varepsilon_0)]$$

and the integral equation

$$(2.4.25)\quad g(Q,R;\varepsilon) = \gamma(Q,R;\varepsilon,\varepsilon_0) + \iiint_D L_\varepsilon[\gamma(P,R;\varepsilon,\varepsilon_0)]g(P,Q;\varepsilon)d\tau_P$$

for the unknown $g(Q,R;\varepsilon)$. This integral equation is of the Fredholm type and admits resolution by a Neumann's series.

We might have chosen $s(P,Q;\varepsilon)$ to depend even analytically on ε. This choice is of interest in the case that the coefficient $p(x_i)$ of (1.1) depends analytically on its variables. In fact, we can then assert that $K^{(\varepsilon,\varepsilon_0)}$ depends analytically on ε and we have the following result: If $p(x_i)$ is an analytic function of the x_i's, the Green's function $g(P,R;\varepsilon)$ depends analytically upon its parameter ε.

Under our more general assumptions, we can only assert that $g(Q,R;\varepsilon)$ is continuously differentiable with respect to ε. For this purpose, we have to assume only that the parametrix depends differentiably on ε. We divide the identity (4.20) by $(\varepsilon-\varepsilon_0)$ and pass to the limit $\varepsilon=\varepsilon_0$. We see immediately that the partial derivative of $g(Q,R;\varepsilon)$ with respect to the parameter ε exists and satisfies the equation

$$(2.4.26)\quad \frac{\partial g(Q,R;\varepsilon)}{\partial\varepsilon} = \frac{\partial s(Q,R;\varepsilon)}{\partial\varepsilon} + \iiint\limits_D L'_\varepsilon[g(P,R;\varepsilon)]g(P,Q;\varepsilon)d\tau_P$$

$$+ \iiint\limits_D L_\varepsilon\left[\frac{\partial s(P,R;\varepsilon)}{\partial\varepsilon}\right]g(P,Q;\varepsilon)d\tau_P \quad ,$$

where

$$(2.4.27)\quad L'_\varepsilon[u] = \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i}\left[\frac{\partial A_{ik}}{\partial\varepsilon}\,\frac{\partial u}{\partial x_k}\right] - \frac{\partial\rho}{\partial\varepsilon}u \quad .$$

We see also from the integral equation (4.20) that the first partial derivatives of $g(Q,R;\varepsilon)$ depend continuously on ε. Using this fact, we can divide the identity (4.14) by $(\varepsilon-\varepsilon_0)$ and pass to the limit $\varepsilon=\varepsilon_0$; we find

$$(2.4.28)\qquad \frac{\partial g(Q,R;\varepsilon)}{\partial\varepsilon} = -\iiint\limits_D E'_\varepsilon[g(P,Q;\varepsilon),g(P,R;\varepsilon)]d\tau_P \quad ,$$

where $E'_\varepsilon[u,v]$ is the bilinear form

$$(2.4.29)\quad E'_\varepsilon[u,v] = \sum_{i,k=1}^{3} \frac{\partial}{\partial\varepsilon}A_{ik}(x_j;\varepsilon)\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_k} + \frac{\partial\rho(x_j;\varepsilon)}{\partial\varepsilon}uv \quad .$$

Formula (4.28) is not only more symmetric than (4.26), but it has, moreover, the advantage of being independent of the choice of the parametrix. It is a typical variational formula, expressing the change of the function with the parameter in terms of the function itself and its partial derivatives with respect to its arguments. The case $\varepsilon = 0$ is of particular interest for most applications and will be considered in detail later on.

5. <u>Some inequalities</u>.

We return to the integral equation (4.20) and observe that the resolution by a Neumann's series leads to a development of $g(P,R;\varepsilon)$ in terms of increasing order in $(\varepsilon - \varepsilon_o)$. We obtain thus a numerically convenient formula for determining the Green's functions of near-by domains with an arbitrary degree of approximation.

We want to point out one feature of this development which leads to interesting inequalities in an important special case. Iterating (4.25), we obtain

$$(2.5.1)\quad g(Q,R;\varepsilon) = \gamma(Q,R;\varepsilon,\varepsilon_o) + \iiint_D L_\varepsilon[\gamma(P,R;\varepsilon,\varepsilon_o)]\,\gamma(P,Q;\varepsilon,\varepsilon_o)\,d\tau_P$$

$$+ \iiint_D \iiint_D L_\varepsilon[\gamma(P,R;\varepsilon,\varepsilon_o)]\,L_\varepsilon[\gamma(M,Q;\varepsilon,\varepsilon_o)]\,g(P,M;\varepsilon)\,d\tau_P\,d\tau_M \quad .$$

In the case that the coefficient $p(x_j)$ of the differential equation (1.1) is non-negative, we observed in Section 1 that $G(P,M)$ is a positive-definite kernel; by definition (2.3) the same is then true of $g(P,M;\varepsilon)$. Hence, the last double integral in (5.1) is non-negative and we are led to the inequality

$$(2.5.2)\quad g(Q,R;\varepsilon) \geqslant \gamma(Q,R;\varepsilon,\varepsilon_o) + \iiint_D L_\varepsilon[\gamma(P,R;\varepsilon,\varepsilon_o)]\,\gamma(P,Q;\varepsilon,\varepsilon_o)\,d\tau_P \quad .$$

We can generalize this last inequality as follows. Let Q_ν be a set of N points in D and λ_ν be N arbitrary real numbers. From the positive-definite character of the kernel $g(P,M;\varepsilon)$ and the identity (5.1), we can then derive the inequality

$$(2.5.3) \qquad \sum_{i,k=1}^{N} [g(Q_i,Q_k;\varepsilon) - \gamma(Q_i,Q_k;\varepsilon,\varepsilon_o)]\lambda_i\lambda_k \geqslant \iiint_D L_\varepsilon[h]h\,d\tau \quad ,$$

where

$$(2.5.3') \qquad h(P) = \sum_{i=1}^{N} \gamma(P,Q_i;\varepsilon,\varepsilon_o)\lambda_i \quad .$$

These inequalities become particularly simple if we choose the parametrix $s(P,Q;\varepsilon)$ in such a way that $s(P,Q;\varepsilon_o) = g(P,Q;\varepsilon_o)$. In fact, in this case we have by definition (4.23)

$$(2.5.4) \qquad \gamma(Q,R;\varepsilon,\varepsilon_o) = s(Q,R;\varepsilon) \quad .$$

The inequalities (5.2) and (5.3) are sharp, since we obtain equality if we choose $s(P,Q;\varepsilon) = g(P,Q;\varepsilon)$.

It is not difficult to derive these inequalities in a direct and elementary way, but it should be observed that they are only one set of a great number of estimates provided by the integral equation (4.20) in the case of a positive-definite Green's function. For example, we can iterate (5.1) again and can express $g(Q,R;\varepsilon)$ in terms of known quantities and a remainder integral

$$(2.5.5) \qquad \iiint_D\iiint_D\iiint_D\iiint_D L_\varepsilon[\gamma(P,R)]L_\varepsilon[\gamma(M,Q)]L_\varepsilon[\gamma(T,P)]L_\varepsilon[\gamma(S,M)]g(T,S;\varepsilon)\,d\tau_P d\tau_M d\tau_S d\tau_T$$

which is again non-negative. Iterating the integral equation for $g(Q,R;\varepsilon)$ in this way, we can approximate the unknown function arbitrarily and with a remainder term of non-negative value. Each approximation leads at the same time to an inequality for the desired Green's function.

6. Variational formulas and variational tensors.

Thus far we considered the parameter ε in the transformation (2.2) as a sufficiently small but finite quantity. We shall now obtain considerable simplifications in our formulas if we treat ε as an infinitesimal, that is,

if we retain in all our formulas only terms which are of the first order, at most, in ε. The formulas thus obtained are indeed variational formulas, since they determine the Green's functions of domains which are in an infinitesimal neighborhood of the original domain D, belonging to $\varepsilon = 0$.

We calculate at first the coefficients $A_{ik}(x_j;\varepsilon)$ up to the first order in ε. Since

(2.6.1) $$\frac{\partial x_j^*}{\partial x_i} = \delta_{ij} + \varepsilon \frac{\partial S_j}{\partial x_i} ,$$

we have in the required degree of precision

(2.6.2) $$\theta(x_i) = 1 + \varepsilon \sum_{i=1}^{3} \frac{\partial S_i}{\partial x_i} + o(\varepsilon)$$

and

(2.6.3) $$\frac{\partial x_i}{\partial x_j^*} = \delta_{ij} - \varepsilon \frac{\partial S_i}{\partial x_j} + o(\varepsilon) .$$

Hence, using definitions (2.8) and (2.8'), we find

(2.6.4) $$A_{ik}(x_j;\varepsilon) = \delta_{ik} + \varepsilon \left\{ \delta_{ik} \sum_{j=1}^{3} \frac{\partial S_j}{\partial x_j} - \frac{\partial S_i}{\partial x_k} - \frac{\partial S_k}{\partial x_i} \right\} + o(\varepsilon) ,$$

(2.6.4') $$\rho(x_j;\varepsilon) = p(x_j) + \varepsilon \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (pS_j) + o(\varepsilon) .$$

Now we are ready to derive an elegant variational formula from the identity (4.14). We define the "variational tensor"

(2.6.5) $$T_{ik}(P;Q,R) = \frac{\partial g(P,Q;o)}{\partial x_i} \frac{\partial g(P,R;o)}{\partial x_k} + \frac{\partial g(P,R;o)}{\partial x_i} \frac{\partial g(P,Q;o)}{\partial x_k}$$
$$- \nabla_P g(P,Q;o) \cdot \nabla_P g(P,R;o)\, \delta_{ik} ,$$

and using it, we put (4.14) in the form

(2.6.6) $$g(Q,R;\varepsilon) - g(Q,R;o) = \varepsilon \iiint_D \left\{ \sum_{i,k=1}^{3} T_{ik}(P;Q,R) \frac{\partial S_i(P)}{\partial x_k} \right.$$
$$\left. - \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (pS_k) g(P,Q;o) g(P,R;o) \right\} d\tau_P + o(\varepsilon) .$$

The variational tensor $T_{ik}(P;Q,R)$ will play a central role in the sequel. It is based only on the original Green's function $g(P,Q;o)$ and is to be considered as known. The tensor is symmetric in its indices as well as in its dependence on Q and R. For $Q=R$ the tensor becomes a Maxwell tensor of the type frequently used in electrostatics.

The trace of the tensor is

(2.6.7) $$T(P;Q,R) = \sum_{k=1}^{3} T_{kk}(P;Q,R) = \nabla_P g(P,Q;o) \cdot \nabla_P g(P,R;o) \quad .$$

We verify easily the following identity:

(2.6.8) $$\begin{aligned} \sum_{k=1}^{3} T_{ik}(P;Q,R)T_{jk}(P;Q,R) = \\ \tfrac{1}{2}[T(P;Q,Q)T_{ij}(P;R,R) + T(P;R,R)T_{ij}(P;Q,Q)] \\ - T(P;Q,R)T_{ij}(P;Q,R) + \delta_{ij}T(P;Q,Q)T(P;R,R) \quad , \end{aligned}$$

which reduces for $Q=R$ to

(2.6.9) $$\sum_{k=1}^{3} T_{ik}(P;Q,Q)T_{jk}(P;Q,Q) = \delta_{ij}T(P;Q,Q)^2 \quad .$$

This shows the important fact that the rows and columns of the Maxwell tensor $T_{ik}(P;Q,Q)$ are orthogonal to each other.

So far we used only the formal structure of the variational tensor; now we utilize the fact that the function $g(P,Q;o)$ satisfies the differential equation (1.1) in dependence on each variable. We find then easily

(2.6.10) $$\sum_{k=1}^{3} \frac{\partial T_{ik}}{\partial x_k} = p(x_j) \frac{\partial}{\partial x_i} [g(P,Q;o)g(P,R;o)] \quad .$$

This simple differential identity permits us to bring the variational formula (6.6) into the simple form

$$(2.6.11) \quad g(Q,R;\varepsilon) - g(Q,R;o) = \varepsilon \iiint_D \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left(\sum_{i=1}^{3} [T_{ik} - p(P)g(P,Q;o)g(P,R;o)\,\delta_{ik}] S_i(P) \right) d\tau_P + o(\varepsilon) \quad .$$

We have thus expressed the first variation of $g(Q,R)$ as an integral over the domain D whose integrand is a divergence term. We assumed D to have a smooth boundary surface C and may, therefore, transform this integral into a surface integral over C. We make use of the fact that on C

$$(2.6.12) \qquad g(P,Q;o) = 0 \quad , \quad \nabla_P g(P,Q;o) = \frac{\partial g(P,Q)}{\partial \nu_P} \nu \, , \quad P \in C \quad .$$

We observe further that the components of the tensor $T_{ik}(P;Q,R)$ become infinite for $P = Q$ and $P = R$. Hence, we have to exclude these singularities from the domain of integration by infinitesimal spheres and to take into account their contributions. After an easy calculation, we find

$$(2.6.13) \quad g(Q,R;\varepsilon) - g(Q,R;o) = -\iint_C \frac{\partial g(P,Q;o)}{\partial \nu_P} \frac{\partial g(P,R;o)}{\partial \nu_P} (\varepsilon\, S \cdot \nu) d\sigma$$

$$+ \varepsilon \sum_{k=1}^{3} \left[\frac{\partial g(Q,R;o)}{\partial q_k} S_k(Q) + \frac{\partial g(R,Q;o)}{\partial r_k} S_k(R) \right] + o(\varepsilon) \quad ,$$

where S is the vector field with components S_i and where the q_k's and r_k's are the coordinates of Q and R, respectively.

It becomes now convenient to return to the original Green's function by means of the correspondence (2.3). We denote by

$$(2.6.14) \qquad \varepsilon\, S \cdot \nu = \delta \nu$$

the normal shift of C under the infinitesimal deformation (2.2) and obtain up to higher order terms in ε

$$(2.6.15) \qquad \delta\, G(Q,R) = -\iint_C \frac{\partial G(P,Q)}{\partial \nu_P} \frac{\partial G(P,R)}{\partial \nu_P} \delta \nu_P d\sigma_P \quad .$$

This elegant variational formula was derived by Hadamard [10] in the case of Laplace's equation for any number of independent variables. The more complicated formulas (6.6) and (6.11) have the advantage, however, of being valid also for domains whose boundary surface C is no longer smooth. We shall discuss this extension of the formulas in the next section. Hadamard's formula is frequently used in variational considerations because of its great formal simplicity; it leads often to a heuristic solution of extremum problems which must then be established precisely by a finer technique which can, in general, be based on the formulas of interior variation of the type (6.6).

We have connected in this section Hadamard's variational formula with the theory of the variation of the Green's function in a fixed domain under the variation of a parameter in the coefficients of the equation. This latter theory is essentially due to Hilbert; it provides a simple proof for the Hadamard formula and permits an evaluation of the error term arising from neglect of the higher order terms.

7. Extension of the variational formula.

It is obviously necessary to extend the variational formula (6.6) to the most general domain D in space for which a Green's function exists. In fact, if we are dealing with extremum problems for domains D involving their Green's functions, we will have to characterize the extremum domain by varying it and comparing its Green's function with that of near-by domains. In this way, we will be able to express analytically its extremum property in the form of identities. But we are not sure, a priori, that the boundary surface C of the extremum domain D is smooth; hence, we cannot apply the results of the preceding section without getting rid of the assumption of smoothness on C. We will show now in this section that the variational formula (4.14) holds in the most

general case and that the formula (6.6) which has been derived from it in a formal way is, consequently, always applicable.

Let D be an arbitrary domain in space which possesses a Green's function G(P,Q) with respect to the differential equation (1.1). It is easy to see that the deformations (2.2) carry D into a new domain D^* which will also have a Green's function $G^*(P,Q)$ if ε is small enough. We can then define in D the functions $g(P,Q;\varepsilon)$ as we did in Section 2.

Let D_n be a sequence of domains with smooth boundary surfaces C_n which converge to D in such a way that $D_n \subset D_{n+1} \subset D$. We denote by $g_n(P,Q;\varepsilon)$ the Green's function of D_n corresponding to $g(P,Q;\varepsilon)$. We can define $g_n(P,Q;\varepsilon)$ as a piecewise smooth function in D by putting $g_n(P,Q;\varepsilon) = 0$ if either argument point lies in $D-D_n$. Putting

$$(2.7.1) \qquad u = g(P,Q;\varepsilon) \quad v = g_n(P,R;\varepsilon_o) \quad , \quad Q,R \in D_n \quad ,$$

we can now obviously apply the first Green's identity (4.1), and taking notice of the singularities of u and v, we find

$$(2.7.2) \qquad g_n(Q,R;\varepsilon_o) = Q_\varepsilon[g(P,Q;\varepsilon), g_n(P,R;\varepsilon_o)] \quad .$$

Now, we can pass to the limit $n \to \infty$; because of the well-known continuity property of the Green's function, we obtain

$$(2.7.3) \qquad g(Q,R;\varepsilon_o) = Q_\varepsilon[g(P,Q;\varepsilon), g(P,R;\varepsilon_o)] \quad .$$

This is formula (4.11), valid now for the most general domains which have a Green's function at all. In the same way, we can prove the analogous identity (4.11') and the desired formula (4.14) follows again by subtraction of the two identities. Thus, the variational formula (6.6) has been established in the most general case.

It is to be observed that we are using here the term Green's function in the generalized sense, namely, as the limit of the Green's functions (in the

proper sense) of smoothly bounded exhausting subdomains. In general, it cannot be asserted that the generalized Green's function vanishes at all points of the boundary C of D. But it is necessary to use this concept if we want to establish a useful functional analysis with respect to the differential equation (1.1).

8. Two independent variables.

Our preceding results are in no way restricted to the case of three independent variables, but can be generalized to n variables if the nature of the parametrix singularity is properly adapted. The case $n = 2$ is of particular interest, since the use of complex variables permits various interesting simplifications in the formulas.

We start again with the differential equation

$$(2.8.1) \qquad L[u] = \nabla^2 u - pu = 0 \quad , \quad \nabla^2 = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} \quad ,$$

and we call $G(P,Q)$ the Green's function of (8.1) for a plane domain D if it satisfies the three requirements

a) For fixed $Q \in D$, $G(P,Q)$ is a solution of (8.1) as a function of P.

b) $G(P,Q) + \frac{1}{2\pi} \log r(P,Q)$ is continuous in D, has uniformly bounded first derivatives, except possibly at Q, and its second derivatives grow at most like $r(P,Q)^{-1}$ if P approaches Q.

c) $G(P,Q)$ vanishes for $P \in C$, $Q \in D$, where C is the boundary curve of D.

We introduce again infinitesimal deformations based on a vector field $S_i(x_j)$ $(i,j = 1,2)$ and we introduce Green's functions $g(P,Q;\varepsilon)$ in the same way as in Section 2. We obtain again the variational formula (6.6) with a variational tensor (6.5); the only difference is that the indices i,k range only over 1,2 instead of 1,2,3.

There occur, however, as if by accident, remarkable relations in this special case. We find

$$(2.8.2)\qquad T_{11}(P;Q,R) = \frac{\partial g(P,Q;o)}{\partial x_1}\frac{\partial g(P,R;o)}{\partial x_1} - \frac{\partial g(P,Q;o)}{\partial x_2}\frac{\partial g(P,R;o)}{\partial x_2} = -T_{22}(P;Q,R)$$

$$T_{12}(P;Q,R) = \frac{\partial g(P,Q;o)}{\partial x_1}\frac{\partial g(P,R;o)}{\partial x_2} + \frac{\partial g(P,Q;o)}{\partial x_2}\frac{\partial g(P,R;o)}{\partial x_1} = T_{21}(P;Q,R)$$

Thus, T_{ik} is a symmetric orthogonal tensor with trace zero; we have

$$(2.8.3)\qquad \sum_{k=1}^{2} T_{ik}T_{jk} = \delta_{ij}\,|\nabla_P g(P,Q;o)|^2\,|\nabla_P g(P,R;o)|^2 \quad .$$

We introduce the complex differential operators

$$(2.8.4)\qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \;,\quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \;,\quad z = x_1 + ix_2 \;,$$

and we write (with $P \equiv z$)

$$(2.8.5)\qquad T_{11}(P;Q,R) = \mathrm{Re}\left\{\frac{\partial g(P,Q;o)}{\partial z}\frac{\partial g(P,R;o)}{\partial z}\right\} = -T_{22}(P;Q,R)$$

$$T_{12}(P;Q,R) = -\,\mathrm{Im}\left\{\frac{\partial g(P,Q;o)}{\partial z}\frac{\partial g(P,R;o)}{\partial z}\right\} = T_{21}(P;Q,R) \quad .$$

For the sake of uniformity, we collect also the deformation vector field $S_\nu(x_j)$ into one complex function

$$(2.8.6)\qquad F(z,\bar z) = S_1(x_j) + i\,S_2(x_j) \quad .$$

The points P,Q,R are replaced by their complex coordinates, z, ξ, η, respectively. With these notations,we can cast (6.6) into the complex form

$$(2.8.7)\qquad g(\xi,\eta;\varepsilon) - g(\xi,\eta;o) = \varepsilon\,\mathrm{Re}\left\{\iint_D \left[8\,\frac{\partial g(z,\xi;o)}{\partial z}\frac{\partial g(z,\eta;o)}{\partial z}\frac{\partial F(z,\bar z)}{\partial \bar z} - 2g(z,\xi;o)g(z,\eta;o)\frac{\partial}{\partial z}(rF)\right]d\tau_z\right\} + o(\varepsilon) \quad .$$

We recognize that the variational formula becomes particularly simple if $F = F(z)$ is an analytic function of the complex variable z. In this case $\frac{\partial F}{\partial \bar{z}} = 0$ and (8.7) reduces to

$$g(\xi, \eta; \varepsilon) - g(\xi, \eta; o) = -2\varepsilon \iint_D g(z, \xi; o) g(z, \eta; o) \operatorname{Re}\left\{ \frac{\partial}{\partial z} (pF) \right\} d\tau_z + o(\varepsilon) \quad . \tag{2.8.8}$$

If we assume, in particular, $p \equiv 0$, we find $\delta g(\xi, \eta) = 0$, which is the well-known invariance of the Green's function of Laplace's equation under a conformal mapping.

We can obtain from (8.7) a variational formula for the original Green's function $G(P,Q)$ connected with the fixed differential equation (8.1) and the varying domains D^*. Let $G(P,Q)$ denote the Green's function of the original domain and let

$$\delta G(P,Q) = G^*(P,Q) - G(P,Q) + o(\varepsilon) \tag{2.8.9}$$

denote the first variation of this Green's function, if the domain D is transformed into D^* by a deformation (2.2). Then, we find easily from (8.7) and Taylor's theorem

$$\delta G(\xi, \eta) = \varepsilon \operatorname{Re}\left\{ \iint_D [8 \frac{\partial G(z,\xi)}{\partial z} \frac{\partial G(z,\eta)}{\partial z} \frac{\partial F}{\partial \bar{z}} - 2G(z,\xi) G(z,\eta) \frac{\partial}{\partial z}(pF)] d\tau_z - 2F(\eta, \bar{\eta}) \frac{\partial G(\eta,\xi)}{\partial \eta} - 2F(\xi,\bar{\xi}) \frac{\partial G(\eta,\xi)}{\partial \xi} \right\} . \tag{2.8.10}$$

9. Singular variations in the plane.

We saw in the preceding section that interior variations based on an analytic function $F(z)$ lead to a particularly simple variational formula for the Green's function. In the very important special case of the Laplace

equation in the plane, which is closely related to the theory of analytic functions, the simplification becomes even too great; since the Green's function is invariant under conformal mapping in this case, we find only that the first variation of $g(P,Q;\varepsilon)$ is zero. In order to obtain a more applicable variational theory of conformal mappings and univalent functions, it proves useful to consider a slightly more general class of deformations. These deformations are based on a complex function $F(z,\bar{z})$ which is analytic in the domain D considered, except in a small fixed circle inside of D. The same type of variation is also of interest in the case of more general differential equations, as will be seen from the applications in Chapter IV.

Let z_o be a fixed point in D and let $K(\varepsilon, z_o)$ denote the circle $|z-z_o| \le \sqrt{\varepsilon} = \rho$. We assume ρ so small that $K \subset D$ and define

$$(2.9.1) \quad F(z,\bar{z}) = \begin{cases} \dfrac{1}{6}\dfrac{e^{i\alpha}}{z-z_o}, & \text{for } z \in D-K, \\[2ex] \dfrac{1}{6}\dfrac{e^{i\alpha}}{z-z_o}\left(6\dfrac{|z-z_o|^2}{\rho^2} - 8\dfrac{|z-z_o|^3}{\rho^3} + 3\dfrac{|z-z_o|^4}{\rho^4}\right), & \text{for } z \in K. \end{cases}$$

By easy calculations it can be seen that this deformation field is twice continuously differentiable in D; it yields a one-to-one mapping of D into a schlicht near-by domain. Thus, our general theory of variations applies.

We insert (9.1) into (8.10) and find easily

$$(2.9.2) \quad \delta G(\xi,\eta) = \varepsilon \operatorname{Re}\left\{\frac{4\pi}{3} e^{i\alpha} \frac{\partial G(z_o,\xi)}{\partial z_o}\frac{\partial G(z_o,\eta)}{\partial z_o} - 2F(\xi,\bar{\xi})\frac{\partial G(\xi,\eta)}{\partial \xi} - 2F(\eta,\bar{\eta})\frac{\partial G(\xi,\eta)}{\partial \eta} - \frac{1}{3}e^{i\alpha}\iint_D G(z,\xi)G(z,\eta)\frac{\partial}{\partial z}\left(\frac{p(z,\bar{z})}{z-z_o}\right)d\tau_z\right\}.$$

Let us suppose that the argument points ξ and η lie outside of the circle $K(\varepsilon, z_o)$. Formula (9.2) represents then the first order change of the Green's function with respect to D and for the differential equation (8.1), if the boundary C of D is mapped by means of the analytic function

$$(2.9.3) \qquad f(z) = z + \frac{1}{6}\,\frac{e^{i\alpha}\rho^2}{z - z_o} \quad ,$$

which is univalent on C. By this mapping, C is transformed into a new curve C^* which bounds a new domain D^* and (9.2) gives the change of the Green's function under this variation.

Variations of the type

$$(2.9.3') \qquad z^* = z + \frac{e^{i\alpha}\rho^2}{z - z_o}$$

have been used successfully in the theory of schlicht functions and are called "interior variations" of the domain. The factor $\frac{1}{6}$ which occurs in (9.3) is unessential and has only been introduced in order to permit a simple continuation of the deformation into the critical circle $K(\varepsilon, z_o)$ which remains one-to-one inside this circle. We reformulate our result (9.2) in the following way: Let (9.3') transform the domain D into D^* and map the points ξ and η into ξ^* and η^*. If $G^*(\xi, \eta)$ is the Green's function of the new domain D^*, we have

$$(2.9.4) \quad G^*(\xi^*, \eta^*) - G(\xi, \eta) = \mathrm{Re}\left\{ 8\pi\rho^2 e^{i\alpha}\, \frac{\partial G(z_o, \xi)}{\partial z_o}\, \frac{\partial G(z_o, \eta)}{\partial z_o} - 2\rho^2 e^{i\alpha} \iint_D G(z, \xi) G(z, \eta)\, \frac{\partial}{\partial z}\left(\frac{p(z, \bar{z})}{z - z_o}\right) d\tau_z \right\} + o(\rho^2) \quad .$$

This result is obviously of particular elegance in the case of the Laplace equation ($p \equiv 0$), but is also useful in the more general case. Its main advantage lies in the fact that no derivatives of the Green's function occur under the integral sign.

10. Variational formulas for the Neumann's function.

We derived in Section 6 an interior variational formula for the Green's function by using the identity (4.14) and the fact that $g(P,Q;\varepsilon)$ and its first partial derivatives depend continuously on ε. We can derive in the same way

analogous formulas for various other important domain functions related to the differential equation (1.1).

We define the Neumann's function $N(P,Q)$ for a three-dimensional domain D with respect to the differential equation (1.1) as usual:

a) $N(P,Q)$ is, for fixed $Q \in D$, a solution of (1.1) as a function of P.

b) It has in D and at Q the same differentiability properties as the Green's function.

c) For fixed $Q \in D$ and $P \in C$, we have

$$\frac{\partial N(P,Q)}{\partial \nu_P} = 0 \quad . \tag{2.10.1}$$

Let us assume that for a given domain D such a Neumann's function does exist. We introduce the infinitesimal deformations (2.2) and obtain near-by domains D^* with Neumann's functions $N^*(P,Q)$, if $|\varepsilon|$ is small enough. Let us define in analogy to (2.3) the functions

$$n(P,Q;\varepsilon) = N^*[P^*(P),Q^*(Q)] \tag{2.10.2}$$

in the fixed domain D, and let us study their dependence on ε. We observe that $n(P,Q;\varepsilon)$ is symmetric in P and Q, in view of the well-known symmetry of the Neumann's function; it satisfies in each variable the differential equation (2.9), since $N^*(P,Q)$ satisfies (1.1) in D^*.

In order to establish the conditions which $n(P,Q;\varepsilon)$ fulfills on the boundary surface C, we make the following observation. Let $U^*(P^*)$ be an arbitrary solution of (1.1) in D^* which is continuously differentiable in D^*+C^*; let $V^*(P^*)$ be continuously differentiable in D^*+C^*. We define in D the functions

$$u(P) = U^*[P^*(P)] \quad , \quad v(P) = V^*[P^*(P)] \tag{2.10.3}$$

and find by (2.6) and (2.10)

$$(2.10.4)\qquad \iiint_D Q_\varepsilon[u,v]d\tau = \iiint_{D^*} [\nabla U^* \cdot \nabla V^* + pU^* V^*]d\tau^* \quad .$$

We use now Green's first identity on both sides of (10.4) and observe that u satisfies in D the equation (2.9), while U^* satisfies in D^* equation (1.1). Thus, we obtain

$$(2.10.5)\qquad \iint_C vP_\varepsilon[\nu, \nabla u]d\sigma = \iint_{C^*} V^* \frac{\partial U^*}{\partial \nu^*} d\sigma^* \quad .$$

Since v(P) is arbitrary, we obtain from this relation the identity

$$(2.10.6)\qquad P_\varepsilon[\nu, \nabla u]d\sigma = \frac{\partial U^*}{\partial \nu^*} d\sigma^* \quad ,$$

which clarifies the meaning of the important linear functional P_ε. We obtain, in particular, the following boundary condition for the function $n(P,Q;\varepsilon)$:

$$(2.10.7)\qquad P_\varepsilon[\nu, \nabla_P n(P,Q;\varepsilon)] = 0 \quad , \quad \text{for } P\in C,\ Q\in D \quad .$$

If we now apply (4.1) with respect to two functions

$$(2.10.8)\qquad u = n(P,Q;\varepsilon) \quad , \quad v = n(P,R;\varepsilon_o) \quad , \quad Q,R\in D \quad ;$$

we obtain by the same calculations as were already performed in Section 4

$$(2.10.9)\qquad n(Q,R;\varepsilon_o) = Q_\varepsilon[n(P,Q;\varepsilon), n(P,R;\varepsilon_o)] \quad .$$

Interchanging ε with ε_o and subtracting the resulting formula from (10.9), we prove finally the identity

$$(2.10.10)\qquad n(Q,R;\varepsilon) - n(Q,R;\varepsilon_o) = -\iiint_D E^{(\varepsilon,\varepsilon_o)}[n(P,Q;\varepsilon), n(P,R;\varepsilon_o)]d\tau_P \quad .$$

We do not enter here into the detailed proof that $n(Q,R;\varepsilon)$ and its first partial derivatives depend continuously upon the parameter ε. We proceed immediately to derive from (10.10) in a formal way the variational formula for $N(Q,R) = n(Q,R;o)$. We define the variational tensor for the Neumann's function to be

$$(2.10.11) \qquad \tilde{T}_{ik}(P;Q,R) = \frac{\partial N(P,Q)}{\partial x_i}\frac{\partial N(P,R)}{\partial x_k} + \frac{\partial N(P,Q)}{\partial x_k}\frac{\partial N(P,R)}{\partial x_i} - \nabla_P N(P,Q)\cdot\nabla_P N(P,R)\,\delta_{ik} \quad ,$$

and we derive from (10.10) the result

$$(2.10.12) \qquad n(Q,R;\varepsilon) - N(Q,R) = \varepsilon\iiint_D \left\{ \sum_{i,k=1}^{3} \tilde{T}_{ik}(P;Q,R)\frac{\partial S_i(P)}{\partial x_k} - \sum_{k=1}^{3}\frac{\partial}{\partial x_k}(pS_k)N(P,Q)N(P,R)\right\} d\tau_P + o(\varepsilon) \quad .$$

This is the fundamental interior variational formula for the Neumann's function, which stands in complete analogy to formula (6.6).

The tensor $\tilde{T}_{ik}$ satisfies the same differential relation as the variational tensor T_{ik}, namely

$$(2.10.13) \qquad \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\tilde{T}_{ik}(P;Q,R) = p(P)\frac{\partial}{\partial x_i}(N(P,Q)N(P,R)) \quad ;$$

thus, we may put (10.12) into the form

$$(2.10.14) \qquad n(Q,R;\varepsilon) - N(Q,R) = \varepsilon\iiint_D \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\sum_{i=1}^{3}[\tilde{T}_{ik} - p(P)N(P,Q)N(P,R)\,\delta_{ik}]\cdot S_i(P)\right)d\tau_P + o(\varepsilon) \quad .$$

Following through the same calculations as in Section 6, we obtain, by integration and use of the boundary condition $\frac{\partial N}{\partial \nu} = 0$ on C, the final formula

$$(2.10.15) \quad \delta N(Q,R) = \iint_C [\nabla_P N(P,Q)\cdot\nabla_P N(P,R) + p(P)N(P,Q)N(P,R)]\,\delta\nu_P\, d\sigma_P \, .$$

This formula corresponds to the Hadamard variational formula (6.15) for the Green's function and plays a similar role.

We want to remark finally that the reasoning of this section can easily be extended to the so-called Robin's functions of a domain D with respect to the differential equation (1.1). A Robin's function is defined like the

Neumann's function and the Green's function; but instead of the boundary conditions $G = 0$ or $\frac{\partial N}{\partial \nu} = 0$, we have for a given positive function μ on C

$$(2.10.16) \qquad \frac{\partial R(P,R)}{\partial \nu_P} = \mu(P)R(P,Q) \quad , \quad \text{for } P \in C, \quad Q \in D \quad .$$

In order to carry out the preceding reasoning in the simplest way, it is convenient to vary the coefficient $\mu(P)$ together with the domain D in such a way that the function

$$(2.10.17) \qquad r(P,Q;\varepsilon) = R^*[P^*(P),Q^*(Q)]$$

satisfies on the boundary surface C the condition

$$(2.10.18) \qquad P_\varepsilon[\nu,\nabla_P r(P,Q;\varepsilon)] = \mu(P)r(P,Q;\varepsilon) \quad .$$

In this case, it is easily seen that the identity (4.14) holds also for $r(P,Q;\varepsilon)$. Hence we can obtain the same formalism as before. On the other hand, it is not difficult to derive variational formulas for the Robin's function of a fixed domain and a fixed differential equation, but with varying coefficient $\mu(P)$. In this way, the most general variational result may be obtained.

11. The harmonic Neumann's function.

The results of the preceding section are not immediately applicable to the case of the Laplace equation

$$(2.11.1) \qquad \nabla^2 u = 0 \quad ,$$

since in this case no finite domain possesses a Neumann's function in the above sense.

It is customary to define the Neumann's function $N(P,Q)$ in the harmonic case by the following four requirements:

a) $N(P,Q)$ is harmonic for $P \in D$, except for a pole at Q.

b) $N(P,Q) - \frac{1}{4\pi} r(P,Q)^{-1}$ is harmonic at Q.

c) For $P \in C$, $Q \in D$, $N(P,Q)$ has a constant normal derivative.

d) We have the relation

$$\iint_C N(P,Q)\,d\sigma_P \equiv 0 \quad , \quad Q \in D \quad . \tag{2.11.2}$$

The last condition is necessary if we want to ensure the symmetry of the Neumann's function in dependence on its two argument points. We have by Green's theorem the identity

$$\iint_C \frac{\partial N(P,Q)}{\partial \nu_P}\,d\sigma_P = 1 \quad , \tag{2.11.3}$$

and, since the Neumann's function has a constant normal derivative for fixed Q, we can sharpen condition c) to the statement

$$\frac{\partial N(P,Q)}{\partial \nu_P} = A(C)^{-1} \quad , \quad A(C) = \text{surface area of } C. \tag{2.11.4}$$

This result shows that the constant value of the normal derivative is independent of Q and that for any two points Q and R

$$\frac{\partial}{\partial \nu_P}\,[N(P,Q) - N(P,R)] = 0 \quad , \quad P \in C, \quad Q,R \in D \quad . \tag{2.11.5}$$

The function $N(P;Q,R) = N(P,Q) - N(P,R)$ is the nearest approach to the general Neumann's function concept of the last section. It has only the inconvenience that we need for its definition the normalization (11.2) for its components, which cannot be extended to domains with more complicated boundary surfaces.

We introduce, therefore, the function

$$\begin{aligned} N(P,W;Q,V) &= N(P;Q,V) - N(W;Q,V) \\ &= N(P,Q) + N(W,V) - N(P,V) - N(W,Q) \\ &= N(Q,V;P,W) \quad , \end{aligned} \tag{2.11.6}$$

which is symmetric in argument and parameter points and satisfies the following four conditions:

a) $N(P,W;Q,V)$ is harmonic in P and W, except at Q and V.

b) $N(P,W;Q,V) - \frac{1}{4\pi}\,[r(P,Q)^{-1} + r(W,V)^{-1} - r(P,V)^{-1} - r(W,Q)^{-1}]$ is regular and harmonic at Q and V.

c) $N(P,W;Q,V)$ has for fixed $Q,V \in D$ the normal derivative zero on the boundary surface C.

d) $N(P,P;Q,V) \equiv 0$.

It is easily seen that these four conditions determine $N(P,W;Q,V)$ in a unique way and may be taken as its definition. Most formulas of potential theory become more elegant if this function is used. In the case of two independent variables, the corresponding expression is closely related to the symmetric integral of the third kind on a closed Riemann surface, obtained by completing the domain D considered through addition of its double.

We might obviously define a function $N(P,W;Q,V)$ by

$$N(P,W;Q,V) = N(P,Q) + N(W,V) - N(P,V) - N(W,Q) \tag{2.11.6'}$$

also in the case of the general differential equation (1.1), although its role is rather unimportant in this case. Let

$$n(P,W;Q,V;\varepsilon) = N^*(P^*,W^*;Q^*,V^*) \tag{2.11.7}$$

be the corresponding solution of (2.9) obtained by referring the function N^* of the varied domain D^* back to D. We introduce the function

$$N(P;Q,R) = N(P,Q) - N(P,R) \tag{2.11.8}$$

and the variational tensor

$$\widetilde{T}_{ik}(P;R,W;Q,V) = \widetilde{T}_{ik}(P;R,Q) + \widetilde{T}_{ik}(P;W,V) - \widetilde{T}_{ik}(P;R,V) - \widetilde{T}_{ik}(P;W,Q) \tag{2.11.9}$$

$$= \frac{\partial}{\partial x_i} N(P;R,W)\,\frac{\partial}{\partial x_k} N(P;Q,V) + \frac{\partial}{\partial x_k} N(P;R,W)\,\frac{\partial}{\partial x_i} N(P;Q,V)$$

$$- \nabla_P N(P;R,W)\cdot \nabla_P N(P;Q,V)\,\delta_{ik} \quad .$$

We then deduce immediately from (10.12) the result

(2.11.10) $$n(R,W;Q,V;\varepsilon) - N(R,W;Q,V) =$$

$$\varepsilon\iiint_D \left\{ \sum_{i,k=1}^{3} \tilde{T}_{ik}(P;R,W;Q,V)\frac{\partial S_i}{\partial x_k} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (pS_k)N(P;R,W)N(P;Q,V)\right\} d\tau_P + o(\varepsilon) .$$

In this formula, we may pass to the limit $p \equiv 0$ because of the existence of a Neumann's function $N(R,W;Q,V)$ in the case of the Laplace equation. We then obtain

(2.11.11) $$n(R,W;Q,V;\varepsilon) - N(R,W;Q,V) =$$

$$\varepsilon\iiint_D \sum_{i,k=1}^{3} \tilde{T}_{ik}(P;R,W;Q,V) \frac{\partial S_i(P)}{\partial x_k} d\tau_P + o(\varepsilon) .$$

This formula is the basic interior variational formula for the Neumann's function of Laplace's equation. It may be extended to the most general domains D for which the usual generalized Neumann's function exists.

We derived in this section the variational formula (11.11) by a limiting process from the corresponding formula in the general case (1.1). It is obvious that this procedure was only followed for the sake of brevity; one may derive (11.11) directly by the preceding method without such limit considerations.

We may derive from (11.11) and the equation

(2.11.12) $$\sum_{k=1}^{3} \frac{\partial}{\partial x_k} \tilde{T}_{ik}(P;R,W;Q,V) = 0$$

the Hadamard type variational formula

(2.11.13) $$\delta N(R,W;Q,V) = \iint_C \nabla_P N(P;R,W) \cdot \nabla_P N(P;Q,V)\, \delta\nu_P d\sigma_P .$$

It should be observed that for arbitrary $T \in D$, we have

(2.11.14) $$\nabla_P N(P,T;R,W) = \nabla_P N(P;R,W) ,$$

so that the formula (11.13) might as well be formulated in terms of the symmetric Neumann's function $N(P,T;R,W)$ alone.

12. Capacity and virtual mass.

We shall now apply the results of the preceding sections in order to derive variational formulas for some quantities of physical significance. We consider the case of the Laplace equation and put

$$(2.12.1) \qquad G(P,Q) = \frac{1}{4\pi}\left[\frac{1}{r(P,Q)} - g(P,Q)\right] \quad .$$

Since $\frac{1}{4\pi r} - G(P,Q)$ is regular harmonic in the domain D and positive on its boundary surface C, we have by the maximum principle

$$(2.12.2) \qquad g(P,Q) \geqq 0 \quad .$$

The quantity

$$(2.12.3) \qquad C(Q) = g(Q,Q)$$

is called the capacity constant of the domain D (or its boundary surface C) with respect to the point Q. Its negative, $-C(Q)$, is the potential at Q of the charge distribution which is induced on the conducting surface C by a unit charge placed at the point Q. We obtain from Hadamard's formula (6.15)

$$(2.12.4) \qquad \delta C(Q) = 4\pi \iint_C \left(\frac{\partial G(P,Q)}{\partial \nu_P}\right)^2 \delta\nu_P d\sigma_P \quad .$$

This formula shows the monotonic dependence of the capacity constant upon the domain and can be used to obtain a clear insight into the dependence of $C(Q)$ on the boundary C. Of course, one can find also an expression for $\delta C(Q)$ in terms of a domain integral involving the tensor T_{ik}. Since this formula is not very simple, we restrict ourselves to one important special case.

We suppose that the domain D is the exterior of a surface C. The formulas of Section 6 are also valid in this case. If Q lies near infinity, we may develop the Green's function in terms of the coordinates q_i of Q to obtain

$$(2.12.5) \qquad G(P,Q) = \frac{G(P)}{\rho} + O\left(\frac{1}{\rho^2}\right) \quad , \quad \rho = \sqrt{q_1^2 + q_2^2 + q_3^2} \quad .$$

Here, $G(P)$ is a harmonic function of P in D except at infinity; it vanishes for $P \in C$. It is easily seen that $G(P)$ has near infinity a development

$$G(P) = \frac{1}{4\pi}\left[1 - \frac{C(\infty)}{r} + O\left(\frac{1}{r^2}\right)\right] \quad . \tag{2.12.6}$$

$C(\infty)$ is called the capacity of the surface C. We may interpret $G(P)$ as the Green's function for a point charge at infinity; this notion can also be justified by considering the transformation of $G(P)$ under an inversion by reciprocal radii.

We apply now Hadamard's formula (6.15) and derive by comparison of coefficients near infinity the variation of $G(P)$. We find

$$\delta G(P) = -\iint_C \frac{\partial G(P,T)}{\partial \nu_T} \frac{\partial G(T)}{\partial \nu_T} \delta \nu_T \, d\sigma_T \quad , \tag{2.12.7}$$

and using (12.6) we arrive finally at [15]

$$\delta C(\infty) = 4\pi \iint_C \left(\frac{\partial G(T)}{\partial \nu_T}\right)^2 \delta \nu_T d\sigma_T \quad . \tag{2.12.8}$$

Comparing (12.4) with (12.8), we recognize the close analogy between the functionals $C(Q)$ and $C(\infty)$.

We can transform (12.8) by introducing the tensor

$$T_{ik} = 2 \frac{\partial G}{\partial x_i} \frac{\partial G}{\partial x_k} - \delta_{ik}(\nabla G)^2 \quad , \quad G = G(P) \quad . \tag{2.12.9}$$

We verify easily that

$$\sum_{k=1}^{3} \frac{\partial T_{ik}}{\partial x_k} = 0 \quad , \quad i = 1,2,3, \tag{2.12.9'}$$

and we obtain the formula

$$\delta C(\infty) = -4\pi \varepsilon \iiint_D \sum_{i,k=1}^{3} T_{ik} \frac{\partial s_i}{\partial x_k} \, d\tau \tag{2.12.10}$$

for the variation of the capacity under a deformation (2.2). This result could have been deduced directly from (6.6) and is applicable for more general domains than is the case for (12.8)

Let us specialize the deformation vector field and put

$$(2.12.11) \qquad S_1 = x_1 \quad , \quad S_2 = x_2 \quad , \quad S_3 = 0 \qquad ;$$

this deformation leads to a stretching in the ratio $(1+\varepsilon):1$ perpendicular to the x_3-axis. We obtain from (12.10)

$$(2.12.12) \qquad \delta C(\infty) = 8\pi\varepsilon \iiint_D \left(\frac{\partial G}{\partial x_3}\right)^2 d\tau \qquad ,$$

which proves that the capacity of each surface C increases under such a stretching. This result is not obvious if the surface C is not convex, and it cannot be read off from (12.8). This example shows the usefulness of transforming variational formulas into different shapes; for certain types of variational kinematics, monotonicity properties become obvious which are otherwise hidden.

The above result can easily be extended to more general affine transformations. Let $((a_{ik}))$ be a matrix such that

$$(2.12.13) \qquad \sum_{i,k=1}^{3} a_{ik}\alpha_i\alpha_k \leq \frac{1}{2} \sum_{i=1}^{3} a_{ii} \sum_{i=1}^{3} \alpha_i^2 \qquad .$$

Then, the deformations based on the linear vector field

$$(2.12.13') \qquad S_i = \sum_{k=1}^{3} a_{ik}x_k \quad , \quad i = 1,2,3,$$

will obviously lead to an increase of the capacity C.

Let us study next the Neumann's function for the exterior D of a closed surface C. In the case of such an infinite domain, it is possible to ask for a Neumann's function with vanishing normal derivatives on C. By this requirement and the condition of regularity at infinity, N(P,Q) is uniquely determined and can be shown to be symmetric in P and Q. It is to be expected that this Neumann's function will have a much simpler variational behavior than the

harmonic Neumann's functions of finite domains. Indeed, the formulas (10.12) and (10.15) can now be applied immediately, just as in the case $p > 0$. We have

$$(2.12.14) \qquad \delta N(Q,R) = \iint_C \nabla_P N(P,Q) \cdot \nabla_P N(P,R)\, \delta \nu_P d\sigma_P$$

and

$$(2.12.15) \qquad \left.\frac{\partial n(Q,R;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \iiint_D \sum_{i,k=1}^{3} T_{ik}(P;Q,R) \frac{\partial s_i(P)}{\partial x_k} d\tau_P \quad .$$

For Q near infinity, we may develop $N(P,Q)$ as follows in terms of the coordinates q_i of Q:

$$(2.12.16) \qquad N(P,Q) = \frac{1}{4\pi}\left[\frac{1}{\rho} + \frac{1}{\rho^3}\sum_{i=1}^{3} \varphi_i(P) q_i + O(\rho^{-3})\right] \quad , \quad \rho = \sqrt{\sum_{i=1}^{3} q_i^2} \quad .$$

Here the $\varphi_i(P)$ are harmonic functions of P in D except at infinity; they possess near infinity a development

$$(2.12.17) \qquad \varphi_i(P) = x_i + r^{-3} \sum_{k=1}^{3} \alpha_{ik} x_k + O(r^{-3})$$

and satisfy on the boundary C of D the conditions

$$(2.12.18) \qquad \frac{\partial \varphi_i(P)}{\partial \nu} = 0 \quad , \quad P \in C \quad .$$

The function $\varphi_i(P)$ may be interpreted as the velocity potential of an irrotational incompressible fluid flow in the region D, bounded by the rigid wall C, which has at infinity unit velocity in the direction x_i. The coefficient matrix $((\alpha_{ik}))$ can be shown to represent a tensor; it is customary in hydrodynamics to consider the tensor [19]

$$(2.12.19) \qquad w_{ik} = 4\pi \alpha_{ik} - \delta_{ik} V \quad ,$$

where V is the volume enclosed by C. We call $((w_{ik}))$ the tensor of virtual mass of the surface C; it plays an important role in the hydromechanics of the surface C moving in an infinite incompressible fluid.

We derive from (12.14) and (12.16) by a comparison of coefficients of $q_i \rho^{-3}$ the variational formula for the velocity potentials

$$(2.12.20) \qquad \delta\varphi_i(R) = \iint_C \nabla\varphi_i(P)\cdot\nabla_P N(P,R)\,\delta\nu_P d\sigma_P \quad ,$$

and, analogously, we obtain from this formula and (12.17) by a further comparison of coefficients

$$(2.12.21) \qquad \delta\alpha_{ik} = \frac{1}{4\pi}\iint_C \nabla\varphi_i\cdot\nabla\varphi_k\,\delta\nu_P d\sigma_P \quad .$$

These results could also have been derived, of course, from (12.15). We would have found, at first, the variations of φ_i and the α_{ik} expressed as integrals over D and have deduced (12.20) and (12.21) by integration by parts. Since the variational formulas involving domain integrals have an interest in themselves, we want to give at least the variational formula for the tensor (12.19) of the virtual mass. We define

$$(2.12.22) \qquad t^{(i,k)}_{\ell m} = \frac{\partial\varphi_i}{\partial x_\ell}\frac{\partial\varphi_k}{\partial x_m} + \frac{\partial\varphi_i}{\partial x_m}\frac{\partial\varphi_k}{\partial x_\ell} - \delta_{\ell m}(\nabla\varphi_i\cdot\nabla\varphi_k) \quad ,$$

which is for fixed i,k a symmetric tensor in ℓ and m. It is easily verified that its divergence vanishes,

$$(2.12.23) \qquad \sum_{m=1}^{3}\frac{\partial t^{(i,k)}_{\ell m}}{\partial x_m} = 0 \quad .$$

Consider now a variation (2.2), but assume that the functions $S_i(P)$ vanish at infinity at least like r^{-3}; in this case the integrals of $\frac{\partial S_i}{\partial x_k}$, extended over the infinite domain D, converge. Under such a variation, we may replace (12.21) by

$$(2.12.24) \qquad \delta\alpha_{ik} = \frac{\varepsilon}{4\pi}\iiint_D \sum_{\ell,m=1}^{3} t^{(i,k)}_{\ell m}\,\frac{\partial S_\ell}{\partial x_m}\,d\tau \quad .$$

We find similarly for the variation of the volume V

(2.12.25) $$\delta V = -\varepsilon \iiint_D \sum_{\ell=1}^{3} \frac{\partial s_\ell}{\partial x_\ell}\, d\tau \quad ,$$

and consequently, by definition (12.19),

(2.12.26) $$\delta w_{ik} = \varepsilon \iiint_D \sum_{\ell,m=1}^{3} (t_{\ell m}^{(i,k)} + \delta_{ik}\, \delta_{\ell m}) \frac{\partial s_\ell}{\partial x_m}\, d\tau \quad .$$

This formula is of value in the case of extremum problems involving the virtual mass if it is not sure a priori that the boundary surface C of the extremal domain D is regular enough so that the variational formula (12.21) applies. Since extremum problems of this type play an important role in the theory of discontinuity surfaces in fluid dynamics, the interest of the above formula is obvious.

If we study the development (12.5) of the Green's function near infinity in more detail, we are led to a series

(2.12.27) $$G(P,R) = \frac{G(P)}{\rho} + \frac{1}{4\pi\rho^3} \sum_{i=1}^{3} \psi_i(P) q_i + O(\rho^{-3}) \quad ,$$

where the $\psi_i(P)$ are harmonic functions for $P \in D$, except at infinity. There, they have a development

(2.12.28) $$\psi_i(P) = x_i + \frac{\beta_i}{r} + r^{-3} \sum_{k=1}^{3} \varepsilon_{ik}\, x_k + O(r^{-3}) \quad ,$$

and they vanish on the boundary surface C. They form the electrostatic counterpart to the velocity potentials $\varphi_i(P)$ derived above and a completely dual variational theory can be given for them. This theory was developed in the Hadamard notation in [19] and can be readily transformed into the domain integral form involving variational tensors.

13. An extremum problem for the virtual mass.

We shall now apply (12.24) to a modification of the extremum problem in potential theory discussed in Section 2 of Chapter I and shall illustrate thereby the value of interior variational formulas in such extremum questions. We consider a closed surface C_o which bounds a finite body B_o and which is homotopic to a torus. We shall seek another surface C_1 spanned through B_o so that the surface C_o+C_1 has an exterior without irreducible closed curves, and such that the coefficient $\alpha = \alpha_{11}$ connected with the virtual mass of C_o+C_1 is a minimum. We are not able to discuss here the existence of an extremal surface, but we shall concentrate on the necessary conditions for such a minimum.

We define a deformation vector field $S_i(P)$ as follows. We choose a point X not on C_o and describe around X a sphere $K_\rho(X)$ of radius ρ; we suppose ρ so small that K_ρ does not intersect the surface C_o. We determine next a function H(P) which is twice continuously differentiable in the entire space, has the constant value 1 in the sphere $K_{\rho/2}(X)$ and vanishes outside of the sphere K_ρ. Let Q be an arbitrary, but fixed, point in $K_{\rho/2}$, not on C_1; we then introduce the deformation vector field

(2.13.1) $$S_i(P) = \delta_{ij} \frac{1}{r} H(P) \quad , \quad r = \overline{PQ} \quad , \quad j \text{ fixed} \quad .$$

This field has a singularity at the point Q, but it is easily seen that the variational formula (12.24) still holds, although the integrals involved are now improper. The variation (2.2) based on the vector field (13.1) will transform the surface C_o+C_1 into a new surface $C_o+C_1^*$ which is a competing surface with respect to our extremum problem. Hence, its corresponding functional α^* cannot be less than the α of the surface C_o+C_1, whatever the sign of the parameter ε may be. Hence, we find by (12.24)

$$\iiint_D \sum_{\ell,m=1}^{3} t_{\ell m} \frac{\partial s_\ell}{\partial x_m} d\tau = 0 \quad , \quad t_{\ell m} = t_{\ell m}^{(1,1)} \quad , \tag{2.13.2}$$

and by (13.1)

$$\iiint_D \sum_{m=1}^{3} t_{jm} \frac{\partial}{\partial x_m} [\frac{1}{r} H(P)] d\tau = 0 \quad . \tag{2.13.3}$$

Because of the definition of $H(P)$, the last integral is only to be extended over the intersection of the extremum domain D with the sphere K_ρ around the fixed point X. If $Q \in K_{\rho/2}$, as assumed, we may put (13.3) into the form

$$\sum_{m=1}^{3} \iiint_{K_{\rho/2}} t_{jm} \frac{\partial}{\partial x_m} (\frac{1}{\overline{PQ}}) dt_P = \Phi_j^{(\rho)}(Q) \quad , \tag{2.13.4}$$

where $\Phi_j^{(\rho)}$ is a harmonic function of Q in $K_{\rho/2}$. Clearly, also

$$\sum_{m=1}^{3} \iiint_D t_{jm} \frac{\partial}{\partial x_m} (\frac{1}{\overline{PQ}}) dt_P = \Phi_j(Q) \quad , \quad j = 1,2,3, \tag{2.13.5}$$

represents a harmonic function of Q in $K_{\rho/2}$. Suppose now that the sphere $K_{\rho/2}$ intersects the surface C_1; by our construction, the function $\Phi_j(Q)$ is regular harmonic inside $K_{\rho/2}$ even as the argument point Q moves across C_1. If Q lies away from C_o+C_1, it is clear that $\Phi_j(Q)$ is regular harmonic, since the divergence of t_{jm} vanishes identically in D. Thus, we have proved the following theorem: A necessary extremum condition in our problem is that the integral sums (13.5) be regular harmonic functions of Q outside of the given body B_o.

We may consider this condition as a set of three singular integral equations for the variational tensor t_{jm}. We should expect intuitively that the surface C_1 is a sheet spanned through the torus B_o such that C_o+C_1 encloses the same volume as did C_o alone. This follows from the monotonic dependence of α upon the domain. If we suppose, moreover, that C_1 is a smooth surface, we recognize easily that the extremum conditions on the functions $\Phi_j(Q)$ mean exactly that $(\nabla\varphi_1)^2$ is continuous across the sheet C_1, as was indicated in Chapter I,

Section 2. Thus, (13.5) is an essential intermediate result formulating the decisive extremum condition before the smoothness of the extremum surface is ensured.

It is quite instructive to study the plane problem which corresponds to the above question. We give two closed curves C_o and C_o' in the plane and want to connect them by a curve C_1 such that the exterior D of $C_o+C_o'+C_1$ has a minimum virtual mass, say in the x_1-direction. It is easily seen that the preceding reasoning leads to the extremum condition

$$(2.13.6)\qquad \sum_{m=1}^{2}\iint_D t_{jm}\frac{\partial}{\partial x_m}\log\frac{1}{\overline{PQ}}\,dt_P = \Phi_j(Q)\quad ,$$

where $\Phi_j(Q)$ is regular harmonic in the common exterior D_o of C_o and C_o', and

$$(2.13.7)\qquad t_{jm} = 2\frac{\partial\varphi_1}{\partial x_j}\frac{\partial\varphi_1}{\partial x_m} - (\nabla\varphi_1)^2\,\delta_{jm}\quad ,$$

$\varphi_1(P)$ being the velocity potential in the direction of the x_1-axis for D.

In order to solve the set of integral equations (13.6), we differentiate this equation with respect to q_ℓ and use the well-known formulas for the interchange of differentiation and integration in the improper integrals of potential theory. We obtain

$$(2.13.8)\qquad -\pi t_{j\ell} - \sum_{m=1}^{2}\iint_D t_{jm}\frac{\partial^2}{\partial x_\ell\,\partial x_m}\left(\log\frac{1}{\overline{PQ}}\right)dt_P = \frac{\partial\Phi_j(Q)}{\partial q_\ell}\quad .$$

Observe now that, by definition (13.7), we have

$$(2.13.9)\qquad t_{11}+t_{22} = 0\quad ,\quad t_{12} = t_{21}\quad .$$

Using further the fact that $\nabla^2\log\frac{1}{\overline{PQ}} = 0$, we then find

$$(2.13.10)\qquad \frac{\partial\Phi_1}{\partial q_1} - \frac{\partial\Phi_2}{\partial q_2} = \pi(t_{22}-t_{11}) = -2\pi t_{11}\quad ,$$

$$\frac{\partial\Phi_1}{\partial q_2} + \frac{\partial\Phi_2}{\partial q_1} = -\pi(t_{12}+t_{21}) = -2\pi t_{12}\quad .$$

Thus we succeed in the plane case to get rid of the improper integral terms by considering a proper combination of derivatives of the harmonic functions $\Phi_j(Q)$. The new formulas make it evident that the tensor components t_{ik} are harmonic functions in the original given domain D_o.

The formulas can be simplified still more by the use of complex notation. We define the complex-valued function

$$F(z,\bar{z}) = \Phi_1(P) + i\,\Phi_2(P) \quad . \tag{2.13.11}$$

Then, the two equations (13.10) can be united to yield

$$\frac{\partial}{\partial \bar{z}} F(z,\bar{z}) = -\pi(t_{11} + it_{12}) = -4\left(\frac{\partial \varphi_1(z)}{\partial \bar{z}}\right)^2 \quad , \tag{2.13.12}$$

using the complex operators (8.4). Since $F(z,\bar{z})$ is a harmonic function and satisfies

$$\frac{\partial^2 F}{\partial z\, \partial \bar{z}} = 0 \quad , \tag{2.13.13}$$

we conclude that $\dfrac{\partial F}{\partial \bar{z}}$ is independent of z and hence is an analytic function of $\bar{z}$ in D_o. Hence, finally, we obtain from (13.12) the extremum condition that

$$\left(\frac{\partial \varphi_1(z)}{\partial z}\right)^2 = \Phi(z) \tag{2.13.14}$$

is a regular analytic function of z in D_o.

The same method applies to many extremum problems of potential theory in the plane, and this fact explains the easy application of interior variations to such problems. The analogy and the difference between the treatment in the plane and in space becomes obvious. The same accident which permits conformal mapping and analytic functions to play a role in the potential theory of the plane is also responsible for the simple results of the variational technique in this case.

CHAPTER III

THEORY OF THE SECOND VARIATION

1. Generalities.

The method of interior variations based on a deformation field (2.2.2) enables us to calculate also higher order variations of Green's functions in a simple way. In fact, it is easily seen that we can obtain from the basic identity (2.4.14) as many derivatives of $g(Q,R;\varepsilon)$ with respect to the parameter ε as the coefficient $p(x_i)$ in (2.1.1) has Hölder continuous partial derivatives with respect to the variables x_i.

We shall deal in this chapter with the theory of the second variation of the Green's function and derive an elegant expression for it. The significance of this result for the treatment of extremum problems and for the general theory of the domain dependence of the Green's function is obvious. In this section, we prepare the way by establishing some useful identities which have an interest in themselves.

We start from the identity in

$$(3.1.1) \qquad L_\varepsilon[g(Q,R;\varepsilon)] = 0 \quad ,$$

differentiate it with respect to ε and put $\varepsilon = 0$. We obtain

$$(3.1.2) \qquad L\left[\left.\frac{\partial g(Q,R;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right] = -L'[g(Q,R;o)] \quad ,$$

where L is the differential operator defined in (2.1.1) and $L' = L'_o$ is defined by (2.4.27) and has, in view of (2.6.4), the form

$$(3.1.3) \quad L'[u] = \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i} \left[\left(\delta_{ik} \sum_{j=1}^{3} \frac{\partial s_j}{\partial x_j} - \frac{\partial s_i}{\partial x_k} - \frac{\partial s_k}{\partial x_i}\right) \frac{\partial u}{\partial x_k}\right]$$

$$- \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (p s_j) u \quad .$$

This rather complicated expression simplifies considerably if u is a solution of (2.1.1). In this case, we have

(3.1.4) $$L'[u] = -L[\sum_{j=1}^{3} \frac{\partial u}{\partial x_j} S_j] \quad .$$

Since $G(Q,R) = g(Q,R;o)$ is a solution of (2.1.1), we derive from (3.1.2) the representation

(3.1.5) $$\left.\frac{\partial g(Q,R;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \sum_{k=1}^{3} [\frac{\partial G(Q,R)}{\partial q_k} S_k(Q) + \frac{\partial G(Q,R)}{\partial r_k} S_k(R)] + H(Q,R) \quad ,$$

where $H(Q,R)$ is a symmetric function of both argument points which satisfies in dependence on each the partial differential equation (2.1.1). It can easily be seen that $H(Q,R)$ is continuously differentiable even for $Q = R$, so that the first right-hand term in (1.5) contains the singularity of $\frac{\partial g}{\partial \varepsilon}$ at the point $Q = R$.

Since $g(Q,R;\varepsilon) \equiv 0$ in ε for $Q \in C$, $R \in D$, we have also

(3.1.6) $$\frac{\partial g(Q,R;\varepsilon)}{\partial \varepsilon} = 0 \quad , \quad \text{for} \quad Q \in C,\ R \in D \quad ,$$

and hence

(3.1.7) $$H(Q,R) = -\sum_{k=1}^{3} \frac{\partial G(Q,R)}{\partial q_k} S_k(Q) \quad , \quad \text{for} \quad Q \in C,\ R \in D \quad .$$

Since H is a solution of (2.1.1), we can express this function in terms of its boundary values (1.7) by means of the Green's function and find

(3.1.8) $$H(Q,R) = -\iint_C \frac{\partial G(P,Q)}{\partial \nu_P} \frac{\partial G(P,R)}{\partial \nu_P} (S \cdot \nu) d\sigma_P \quad .$$

Comparing this result with (2.6.15), we recognize the significance of the regular function $H(Q,R)$. It represents, up to the factor ε, the first variation of the Green's function $G(Q,R)$ under variation of the domain. This function will play an important role in the variational theory.

2. The second variation.

We consider the formula (2.4.28), which holds identically in ε, and differentiate this identity with respect to the parameter. Since the singularity of the function $\frac{\partial g}{\partial \varepsilon}$ is well known by formula (1.5), it is easy to establish the validity of this process. We have to assume, however, that $p(x_i)$ is twice Hölder continuously differentiable with respect to all its variables, since these second order derivatives will occur in our formulas. After differentiation, we put $\varepsilon = 0$ and obtain

$$(3.2.1)\quad \left.\frac{\partial^2 g(Q,R;\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0} = -\iiint_D E'\left[\left.\frac{\partial g(P,Q;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, G(P,R)\right] d\tau_P$$

$$-\iiint_D E'\left[G(P,Q), \left.\frac{\partial g(P,R;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right] d\tau_P - \iiint_D E''[G(P,Q),G(P,R)]\, d\tau_P \quad .$$

Here, we put $E' = E'_0$ and define E'_ε by (2.4.29); analogously, we define

$$(3.2.2)\quad E''_\varepsilon[u,v] = \sum_{i,k=1}^{3} \frac{\partial^2}{\partial \varepsilon^2} A_{ik}(x_j;\varepsilon)\, \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_k} + \frac{\partial^2 \rho(x_j;\varepsilon)}{\partial \varepsilon^2}\, uv$$

and put $E''_0 = E''$.

Consider the expression $\chi(P,Q) = \left.\frac{\partial g(P,Q;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}$ and the identity

$$(3.2.3)\quad \iiint_D E'[\chi(P,Q),G(P,R)]\, d\tau_P = -\frac{4\pi}{3}\chi(R,Q) \sum_{j=1}^{3} \frac{\partial s_j(R)}{\partial r_j}$$

$$-\iiint_D \chi(P,Q)\cdot L'[G(P,R)]\, d\tau_P \quad ,$$

which follows from Green's identity and the fact that $\chi(P,Q)$ vanishes on the boundary C of D. By virtue of (1.2), we may write this identity also in the form

$$(3.2.4)\quad \iiint_D E'[\chi(P,Q),G(P,R)]\, d\tau_P = -\frac{4\pi}{3}\chi(R,Q) \sum_{j=1}^{3} \frac{\partial s_j(R)}{\partial r_j}$$

$$+\iiint_D \chi(P,Q) L_P[\chi(P,R)]\, d\tau_P \quad .$$

Using again Green's identity and the singularity term of χ given by (1.5), we obtain finally

$$(3.2.5) \qquad \iiint_D E'[\chi(P,Q),G(P,R)]d\tau_P = -Q_o[\chi(P,Q),\chi(P,R)] \quad ,$$

where Q_o is the bilinear form belonging to the Dirichlet integral (2.2.10').

The second term on the right side of (2.1) can be treated in the same way, and we obtain therefore finally

$$(3.2.6) \qquad \left.\frac{\partial^2 g(Q,R;\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0} = 2Q_o\left[\frac{\partial g(P,Q;\varepsilon)}{\partial\varepsilon}, \frac{\partial g(P,R;\varepsilon)}{\partial\varepsilon}\right]_{\varepsilon=0}$$

$$- \iiint_D E''[G(P,Q),G(P,R)]d\tau_P \quad .$$

This important formula permits an easy determination of the second variation of the Green's function.

For the sake of completeness, we give here also the expressions for the terms $\frac{\partial^2}{\partial\varepsilon^2}A_{ik}$ and $\frac{\partial^2}{\partial\varepsilon^2}\rho$. These expressions are obtained by straightforward computation from the definitions (2.2.8) and (2.2.8'). We find

$$(3.2.7) \qquad \frac{1}{2}\left.\frac{\partial^2 A_{ik}(x_\ell;\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0} = -\sum_{j=1}^{3}\left[\frac{\partial(s_i,s_j)}{\partial(x_k,x_j)} + \frac{\partial(s_k,s_j)}{\partial(x_i,x_j)}\right]$$

$$+\sum_{j=1}^{3}\frac{\partial s_i}{\partial x_j}\frac{\partial s_k}{\partial x_j} + \delta_{ik}\left[\frac{\partial(s_1,s_2)}{\partial(x_1,x_2)} + \frac{\partial(s_1,s_3)}{\partial(x_1,x_3)} + \frac{\partial(s_2,s_3)}{\partial(x_2,x_3)}\right] \quad ,$$

$$(3.2.7') \qquad \frac{1}{2}\frac{\partial^2\rho(x_\ell;\varepsilon)}{\partial\varepsilon^2} = p(x_\ell)\left[\frac{\partial(s_1,s_2)}{\partial(x_1,x_2)} + \frac{\partial(s_1,s_3)}{\partial(x_1,x_3)} + \frac{\partial(s_2,s_3)}{\partial(x_2,x_3)}\right]$$

$$+\sum_{i,j=1}^{3}\left[\frac{\partial s_i}{\partial x_i}\frac{\partial p}{\partial x_j}s_j + \frac{1}{2}\frac{\partial^2 p}{\partial x_i\,\partial x_j}s_i s_j\right] \quad .$$

We observe next the particular structure of the function $g(Q,R;\varepsilon)$ as defined by (2.2.3). We have the series development

$$(3.2.8) \qquad g(Q,R;\varepsilon) = G^*(Q,R) + \varepsilon \sum_{j=1}^{3} \left(\frac{\partial G^*}{\partial q_i} S_i(Q) + \frac{\partial G^*}{\partial r_i} S_i(R)\right) + \ldots ,$$

where G^* is the Green's function of the domain D^* obtained from D by the deformation (2.2.2). Suppose now that we change the vector field $S_i(x_\ell)$ arbitrarily in D, but keep it unchanged on the boundary C and at the two points Q and R. Under such a change, D^* would remain invariant, and hence $G^*(Q,R)$ would be unaffected. In view of (3.2.8), we would also find $g(Q,R;\varepsilon)$ preserved under this change of the deformation. Hence, we conclude that $g(Q,R;\varepsilon)$ depends only on the values of S_i on C and at the points Q,R. The same holds also for $\frac{\partial^2 g(Q,R;\varepsilon)}{\partial\varepsilon^2}$. Thus, we know a priori that we can transform the volume integrals in (2.6) into surface integrals extended over the boundary C; under the process of integration by parts, the singular points Q and R will yield certain residua which will lead to the special dependence of the left-hand term on these two points. It is possible, in fact, to obtain such a formula by starting from (2.3). In view of the complicated structure of the expressions (2.7) and (2.7') this procedure is, however, rather involved; we proceed, therefore, in a different way.

We assume, at first, that the boundary C of D is sufficiently often differentiable and that the vector field S_i is chosen in such a way that it has at every point of the boundary C the direction of the outward normal. Then, the original domain D will lie in the deformed domain D^* and the Green's function $G^*(P,Q)$ of D^* will be a solution of (2.1.1) everywhere in D, except at the point $P = Q$. For $P \in C$, we have the condition

$$(3.2.9) \quad G^*(P^*,Q) = G^*(P,Q) + \varepsilon \sum_{i=1}^{3} \frac{\partial G^*(P,Q)}{\partial p_i} S_i(P)$$

$$+ \frac{\varepsilon^2}{2} \sum_{i,k=1}^{3} \frac{\partial^2 G^*(P,Q)}{\partial p_i\, \partial p_k} S_i(P) S_k(P) + o(\varepsilon^2) = 0 \quad .$$

Hence, the function

$$(3.2.10) \qquad \Omega(P,Q) = G^*(P,Q) - G(P,Q)$$

is a regular solution of (2.1.1) in D with the boundary values

$$(3.2.9') \quad -\varepsilon \sum_{i=1}^{3} \frac{\partial G^*(P,Q)}{\partial p_i} S_i(P) - \frac{\varepsilon^2}{2} \sum_{i,k=1}^{3} \frac{\partial^2 G^*(P,Q)}{\partial p_i\, \partial p_k} S_i(P) S_k(P) + o(\varepsilon^2)$$

for $P \in C$.

Therefore, we may represent $\Omega(P,Q)$ by means of the Green's function $G(P,Q)$ and the above boundary values. Since S_i has the exterior normal direction, we find

$$(3.2.11) \quad \Omega(Q,R) = +\varepsilon \iint_C \frac{\partial G^*(P,Q)}{\partial \nu_P} |S(P)| \frac{\partial G(P,R)}{\partial \nu_P} d\sigma_P$$

$$- \frac{\varepsilon^2}{2} \iint_C \frac{\partial^2 G(P,Q)}{\partial \nu_P^2} |S(P)|^2 \frac{\partial G(P,R)}{\partial \nu_P} d\sigma_P + o(\varepsilon^2) \quad .$$

We simplify this result by use of the function $H(Q,R)$, defined in Section 1. Observe that $|S| = -(S \cdot \nu)$ and hence by (1.8) and (1.7)

$$(3.2.12) \quad \Omega(Q,R) = \varepsilon H(Q,R) + \varepsilon \iint_C \frac{\partial \Omega(P,Q)}{\partial \nu_P} H(P,R) d\sigma_P$$

$$- \frac{\varepsilon^2}{2} \iint_C \frac{\partial^2 G(P,Q)}{\partial \nu_P^2} \frac{\partial G(P,R)}{\partial \nu_P} (S \cdot \nu)^2 d\sigma_P + o(\varepsilon^2) \quad .$$

Since $\Omega(Q,R)$ is, up to higher order terms in ε, replaceable by $\varepsilon H(Q,R)$, we may write finally

$$(3.2.13) \quad G^*(Q,R) - G(Q,R) = \varepsilon H(Q,R) - \varepsilon^2 Q_o[H(P,Q),H(P,R)]$$

$$- \frac{\varepsilon^2}{2} \iint_C \frac{\partial^2 G(P,Q)}{\partial \nu_P^2} \frac{\partial G(P,R)}{\partial \nu_P} (S \cdot \nu)^2 d\sigma_P + o(\varepsilon^2) \quad .$$

Thus, we have proved under the above restricting assumptions the formula

$$(3.2.14) \quad \delta^2 G(Q,R) = -2\,\varepsilon^2 Q_o[H(P,Q),H(P,R)] - \varepsilon^2 \iint_C \frac{\partial^2 G(P,Q)}{\partial \nu_P^2} \frac{\partial G(P,R)}{\partial \nu_P} (S\cdot\nu)^2\, d\sigma_P \ .$$

In the last integral, we may replace $\frac{\partial^2 G}{\partial \nu^2}$ by lower derivatives if we use the fact that G(P,Q) satisfies the differential equation (2.1.1) and vanishes identically on C. If ρ_1 and ρ_2 denote the principal radii of curvature at the point P, it is easily checked that in view of (2.1.1)

$$(3.2.15) \qquad \frac{\partial^2 G(P,Q)}{\partial \nu_P^2} = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \frac{\partial G(P,Q)}{\partial \nu_P} \ .$$

Hence, we arrive finally at the result

$$(3.2.16) \qquad \delta^2 G(Q,R) = -2\,\varepsilon^2 Q_o[H(P,Q),H(P,R)] - \varepsilon^2 \iint_C \frac{\partial G(P,Q)}{\partial \nu_P} \frac{\partial G(P,R)}{\partial \nu_P} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(S\cdot\nu)^2 d\sigma_P \ .$$

We want next to rid ourselves of the assumption that the deformation vector field S_i has everywhere normal direction with respect to the boundary C. Let us suppose that only the assumption $(\nu\cdot S) < 0$ is fulfilled on C. It is easy to determine the normal shift λ on C which is necessary in order to transform this surface into the surface C^* induced by the deformation (2.2.2). In fact, let $x_i(u,v)$ be a parametric representation of the surface C and let $\nu_i(u,v)$ be the components of the unit vector on C in the direction of the interior normal. We want to represent the point x_i^* of C^* in terms of a normally displaced point $x_i(u+\varepsilon U, v+\varepsilon V)$ on C. Thus, we have for a suitable function λ of $u^* = u + \varepsilon U$ and $v^* = v + \varepsilon V$ the condition

$$(3.2.17) \qquad x_i(u,v) + \varepsilon S_i(u,v) = x_i(u,v) + \varepsilon\left(\frac{\partial x_i}{\partial u} U + \frac{\partial x_i}{\partial v} V\right) + \frac{\varepsilon^2}{2}\left[\frac{\partial^2 x_i}{\partial u^2} U^2 + 2\frac{\partial^2 x_i}{\partial u\,\partial v} UV + \frac{\partial^2 x_i}{\partial v^2} V^2\right] + \varepsilon\lambda(u^*,v^*)\left\{\nu_i(u,v) + \varepsilon\left(\frac{\partial \nu_i}{\partial u} U + \frac{\partial \nu_i}{\partial v} V\right)\right\} + o(\varepsilon^2) \ .$$

Multiply this equation with $\nu_i(u,v)$ and sum over i. Since ν_i is orthogonal to the tangential vectors $\frac{\partial x_i}{\partial u}$, $\frac{\partial x_i}{\partial v}$, $\frac{\partial \nu_i}{\partial u}$, $\frac{\partial \nu_i}{\partial v}$, we obtain

$$(3.2.18) \qquad \lambda(u^*,v^*) = (S\cdot\nu) - \frac{\varepsilon}{2}\,(\mathcal{L}U^2 + 2\mathcal{M}UV + \mathcal{N}V^2) + o(\varepsilon) \quad ,$$

where $\mathcal{L},\mathcal{M},\mathcal{N}$ are the coefficients of the second fundamental form of C. We deduce from (2.17) by comparison of the tangential components

$$(3.2.19) \qquad UE + VF = (S\cdot\frac{\partial x}{\partial u}) + o(1) \ , \ UF + VG = (S\cdot\frac{\partial x}{\partial v}) + o(1) \quad ,$$

where E,F,G are the components of the first fundamental form. Thus, U and V can be expressed linearly in terms of S and geometric quantities of C, up to higher order terms in ε. We obtain finally

$$(3.2.20) \qquad \lambda(u,v) = (S\cdot\nu) - \varepsilon\,[U\,\frac{\partial(S\cdot\nu)}{\partial u} + V\,\frac{\partial(S\cdot\nu)}{\partial v}] - \frac{\varepsilon}{2}\,[\mathcal{L}U^2 + 2\mathcal{M}UV + \mathcal{N}V^2] + o(\varepsilon) \quad .$$

We write for short

$$(3.2.20') \qquad \lambda(u,v) = (S\cdot\nu) - \varepsilon\,\Phi[S] + o(\varepsilon) \quad ,$$

where $\Phi[S]$ is a simple expression of second order in S_i and $\frac{\partial S_i}{\partial x_k}$ which depends on the geometry of the surface C.

Consider now the expression

$$(3.2.21) \quad G^*(Q,R) = G(Q,R) + \delta G(Q,R) + \frac{1}{2}\,\delta^2 G(Q,R) + o(\varepsilon^2) \quad .$$

We have in formulas (2.6.15) and (2.16) simple expressions for the first and second variation if a normal shift $\delta\nu = \varepsilon\,(S\cdot\nu)$ is made. Inserting the value (2.20') into these formulas and collecting the terms of order ε^2, we obtain finally

$$(3.2.22) \qquad \delta^2 G(Q,R) = -2\,\varepsilon^2 Q_o[H(P,Q),H(P,R)] - \varepsilon^2 \iint_C \frac{\partial G(P,Q)}{\partial \nu_P}\,\frac{\partial G(P,R)}{\partial \nu_P}\,[(\frac{1}{\rho_1} + \frac{1}{\rho_2})(S\cdot\nu)^2 - 2\Phi[S]]\,d\sigma \quad .$$

This variational result has been derived under the assumption that $(S\cdot\nu) < 0$ and that the surface C is sufficiently smooth. We know, however, from our general considerations that a formula of this type can be obtained by integration by parts of the expression (2.6). We used the later reasoning only in order to save laborious calculations and to arrive easily at the final result of these transformations. Hence, formula (2.22) must hold for an arbitrary deformation vector field S_1 and for all surfaces C which possess continuous curvature and a continuous second normal derivative of the Green's function. In particular, the elegant formula

$$(3.2.16') \quad \delta^2 G(Q,R) = -2Q_o[\delta G(P,Q), \delta G(P,R)] - \iint_C \frac{\partial G(P,Q)}{\partial\nu}\frac{\partial G(P,R)}{\partial\nu}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\delta\nu^2\, d\sigma$$

holds for all such surfaces under a normal shift $\delta\nu$.

3. Parametrization.

We calculated in the preceding section the second variation of the Green's function of a domain D under a deformation of the particular type (2.2.2). It is necessary to adapt this variational formula to various problems arising in the applications. For example, we may have a family of closed surfaces C_t which depend on a parameter t; they enclose domains D_t with corresponding Green's functions $G_t(Q,R)$. It is necessary to find simple expressions for the first and second derivatives of G_t with respect to the parameter t. These expressions are obtained from our variational formulas by elementary transformations.

Let us suppose that two surfaces C_o and C_1 are given such that C_1 encloses C_o entirely, and let V(P) be a twice continuously differentiable function in the domain bounded by C_1 and C_o. We assume, moreover, that $|\nabla V| \neq 0$ in this region and that the function V(P) has on C_o and C_1 the boundary values 0 and 1,

respectively. Consider now the level surface C_t, defined as the locus

(3.3.1) $$V(P) = t \quad , \quad P \in C_t \quad .$$

These surfaces have no self-intersection and they determine a one-parameter family of finite domains D_t which depend monotonically upon the parameter t. Let $G_t(Q,R)$ be the corresponding Green's function; our problem is to determine the derivatives of G_t with respect to the parameter t.

Let P be a point of C_t and compute the normal shift $\delta\nu$ which is necessary in order to place P on the level surface $C_{t+\Delta t}$. We clearly have the condition

(3.3.2) $$\frac{\partial V}{\partial \nu}\,\delta\nu + \frac{1}{2}\,\frac{\partial^2 V}{\partial \nu^2}\,(\delta\nu)^2 + \ldots = \Delta t \quad ,$$

whence

(3.3.2') $$\delta\nu = \left(\frac{\partial V}{\partial \nu}\right)^{-1} \Delta t - \frac{1}{2}\left(\frac{\partial^2 V}{\partial \nu^2}\right)\left(\frac{\partial V}{\partial \nu}\right)^{-3} \Delta t^2 + o(\Delta t^2) \quad .$$

Therefore, we derive from (2.6.15) and (2.16') the identities

(3.3.3) $$\frac{\partial}{\partial t}\,G_t(Q,R) = -\iint_{C_t} \frac{\partial G_t(P,Q)}{\partial \nu_P}\,\frac{\partial G_t(P,R)}{\partial \nu_P}\left(\frac{\partial V(P)}{\partial \nu}\right)^{-1} d\sigma_P$$

and

(3.3.4) $$\frac{\partial^2}{\partial t^2}\,G_t(Q,R) = -2Q_0\left[\frac{\partial}{\partial t}\,G_t(P,Q)\,,\,\frac{\partial}{\partial t}\,G_t(P,R)\right]$$
$$-\iint_{C_t} \frac{\partial G_t(P,Q)}{\partial \nu_P}\,\frac{\partial G_t(P,R)}{\partial \nu_P}\left(\frac{\partial V(P)}{\partial \nu}\right)^{-2}\left[\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right.$$
$$\left.-\frac{\partial^2 V(P)}{\partial \nu^2}\cdot\left(\frac{\partial V}{\partial \nu}\right)^{-1}\right]d\sigma_P \quad .$$

We observe now that C_t is a level surface for the function V(P). Hence, we have the well-known identity

(3.3.5) $$\nabla^2 V = \frac{\partial^2 V}{\partial \nu^2} - \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\frac{\partial V}{\partial \nu} \quad ,$$

and we may write

$$(3.3.4') \qquad \frac{\partial^2}{\partial t^2} G_t(Q,R) = -2Q_o\left[\frac{\partial}{\partial t} G_t(P,Q), \frac{\partial}{\partial t} G_t(P,R)\right] + \iint_{C_t} \frac{\partial G_t(P,Q)}{\partial \nu_P} \frac{\partial G_t(P,R)}{\partial \nu_P} \left(\frac{\partial V(P)}{\partial \nu}\right)^{-3} \nabla^2 V(P) d\sigma_P \quad .$$

The preceding formulas are particulary interesting in the case of a positive coefficient $p(x_i)$ in the differential equation (2.1.1). In this case, we know that the Green's function is positive in the domain D_t and that $\frac{\partial G_t}{\partial \nu}$ is positive on C_t. Moreover, it is clear that $\frac{\partial V}{\partial \nu}$ is negative on each level surface C_t. Hence, we see that the Green's functions $G_t(Q,R)$ increase monotonically with t by virtue of (3.3), a fact which follows also easily from the minimum principle valid for the Green's function. Let us suppose further that the function V(P) is subharmonic, i.e.,

$$(3.3.6) \qquad \nabla^2 V \geq 0 \quad ,$$

between C_o and C_1. We can then assert that the surface integral in (3.4') is negative and we have the estimate

$$(3.3.7) \qquad \frac{\partial^2}{\partial t^2} G_t(Q,R) \leq -2Q_o\left[\frac{\partial}{\partial t} G_t(P,R), \frac{\partial}{\partial t} G_t(P,R)\right] \quad .$$

Equality will hold in (3.7) in the case of a harmonic function V(P).

In order to draw conclusions from (3.7), we introduce an arbitrary fundamental singularity S(Q,R) for the differential equation (2.1.1) which is defined in the larger domain D_1 bounded by C_1. Thus, the functions

$$(3.3.8) \qquad h_t(Q,R) = G_t(Q,R) - S(Q,R)$$

will be regular solutions of (2.1.1) throughout D_t. We might choose, in particular, $S = G_1$. Clearly, we have

$$(3.3.9) \quad \frac{\partial^2}{\partial t^2} h_t(Q,R) = \frac{\partial^2}{\partial t^2} G_t(Q,R) \leq -2Q_o\left[\frac{\partial}{\partial t} G_t(P,Q), \frac{\partial}{\partial t} G_t(P,R)\right] \quad .$$

Consider now the quadratic form

$$(3.3.10)\qquad \sum_{\rho,\sigma=1}^{N} h_t(Q_\rho,Q_\sigma)\lambda_\rho\lambda_\sigma = A_t$$

based on N arbitrary numbers λ_ρ and N points Q_ρ inside D_o. We conclude from (3.9) and the fact that $Q_o[u] \geqslant 0$ for $p(x_i) \geqslant 0$

$$(3.3.11)\qquad \frac{\partial^2}{\partial t^2} A_t \leqslant 0 \quad ,$$

which shows the concave dependence of the quadratic form A_t upon the parameter t.

This result gives an important and useful insight into the dependence of the Green's function upon the domain D. Consider, for example, the case of Laplace's equation. Here, a fundamental singularity is readily available, namely

$$(3.3.12)\qquad S(Q,R) = \frac{1}{4\pi}\,\frac{1}{r(Q,R)} \quad .$$

We have

$$(3.3.13)\qquad h_t(Q,Q) = -\frac{1}{4\pi} C_t(Q) \quad ,$$

where $C_t(Q)$ is the capacity constant of the domain D with respect to the point Q, as defined in Section 12 of Chapter II. We may therefore state that in the case of a subharmonic function V(P) the capacity constant $C_t(Q)$ is a monotonic and convex function of the parameter t.

Let us return to the case $p(x_i) > 0$. In this case it is well known [1, 2] that there exists a kernel function $K_t(P,Q)$ for the domain D_t and the differential equation (2.1.1) which has the reproducing property

$$(3.3.14)\qquad u(Q) = Q_o[K_t(P,Q),u(P)]$$

for every regular solution U(P) of (2.1.1) in D_t. The Dirichlet integral Q_o is, of course, to be taken over the domain D_t. We have by the Schwarz

inequality and the reproducing property of the kernel the inequality

$$(3.3.15)\qquad u(Q)^2 \leq Q_o[u]\cdot K_t(Q,Q) \quad .$$

Observe now that $\frac{\partial}{\partial t} G_t(P,Q)$ is a regular solution of (2.1.1) in D_t, so that the above inequality can be applied to it. Thus, we deduce from (3.9) the result

$$(3.3.16)\qquad \frac{\partial^2 h_t(Q,Q)}{\partial t^2} \leq -2\left(\frac{\partial h_t(Q,Q)}{\partial t}\right)^2 [K_t(Q,Q)]^{-1} \quad .$$

We may define $[K_t(Q,Q)]^{-1}$ by virtue of (3.15) as the minimum value of the Dirichlet integral $Q_o[u]$ for all solutions $u(P)$ of (2.1.1) in D_t which have at Q the value 1. Hence, this functional increases monotonically with D_t. Thus, (3.16) implies

$$(3.3.17)\qquad \frac{\partial^2 h_t(Q,Q)}{\partial t^2} \leq -2\left(\frac{\partial h_t(Q,Q)}{\partial t}\right)^2 [K_o(Q,Q)]^{-1} \quad ,$$

an inequality involving only the kernel function of the initial domain D_o. Analogous estimates can be obtained, of course, for the more general expressions (3.10).

It is possible to extend the preceding reasoning to the case of Laplace's equation. In particular, if we consider the exterior D of a closed surface C, we can introduce the kernel function

$$(3.3.18)\qquad K(P,Q) = N(P,Q) - G(P,Q) = n(P,Q) + g(P,Q) \quad ,$$

where

$$(3.3.19)\qquad N(P,Q) = \frac{1}{4\pi r(P,Q)} + n(P,Q)\ , \ G(P,Q) = \frac{1}{4\pi r(P,Q)} - g(P,Q)$$

are the Neumann's and the Green's functions of D.

Let C_o be a closed surface including C_1 and let $V(P)$ be subharmonic in the domain between the two surfaces. We assume again that $|\nabla V| \neq 0$, and that $V(P) = 0$ on C_o and $V(P) = 1$ on C_1. Let D_t be the exterior of the level surface

$V = t$ and let G_t be the corresponding Green's function. We find in analogy to (3.9) the inequality

(3.3.20) $$\frac{\partial^2 C_t(\infty)}{\partial t^2} \geq 8\pi Q_o[\frac{\partial}{\partial t} G_t(P)]$$

for the capacity C_t of the domain D_t, where $G_t(P)$ is the Green's function of D_t with the singularity at infinity. Since $\frac{\partial}{\partial t} G_t(P)$ is regular harmonic in D_t, we have

(3.3.21) $$\frac{\partial}{\partial t} G_t(Q) = Q_o[K_t(P,Q), \frac{\partial}{\partial t} G_t(P)]$$

with K_t the kernel function of D_t and, hence, by the Schwarz inequality

(3.3.22) $$[\frac{\partial}{\partial t} G_t(Q)]^2 \leq K_t(Q,Q) Q_o[\frac{\partial}{\partial t} G_t(P)] \quad .$$

Combining (3.20) and (3.22), we find

(3.3.23) $$\frac{\partial^2 C_t(\infty)}{\partial t^2} \geq 8\pi[\frac{\partial}{\partial t} G_t(Q)]^2 K_t[Q,Q]^{-1}$$

for arbitrary $Q \in D_t$. Let now $Q \to \infty$; it can be shown that [19, p. 139]

(3.3.24) $$\lim_{Q\to\infty} r^2 K_t(Q,Q) = \frac{C_t(\infty)}{4\pi} \quad ,$$

while obviously

(3.3.25) $$\lim_{Q\to\infty} r \frac{\partial}{\partial t} G_t(Q) = -\frac{1}{4\pi} \frac{\partial C_t(\infty)}{\partial t} \quad .$$

Hence, we find

(3.3.26) $$\frac{\partial^2 C_t(\infty)}{\partial t^2} \geq 2C_t(\infty)^{-1}(\frac{\partial C_t(\infty)}{\partial t})^2 \quad ,$$

or

(3.3.27) $$\frac{\partial^2}{\partial t^2}\left(\frac{1}{C_t(\infty)}\right) \leq 0 \quad .$$

Thus the reciprocal of the capacity varies as a concave function under a level surface variation with a subharmonic function $V(P)$. This result shows the significance and usefulness of the variational formula (3.4').

We can use (3.27) in discussing various extremum problems. Let, for example, C be a closed surface which encloses a fixed point Q. Define the functional

(3.3.28) $$\pi(C) = C(Q)\cdot C(\infty) \quad ,$$

which is dimensionless, i.e., remains unchanged under a similarity transformation of the surface C.

We call a smooth surface C stationary with respect to the functional π if we have $\delta\pi = 0$ under infinitesimal deformations $\delta\nu$ of C. By the Hadamard formulas (2.12.4) and (2.12.8) this condition has the form

(3.3.29) $$C(Q)^{-\frac{1}{2}} \frac{\partial}{\partial\nu^-} G(P,Q) = C(\infty)^{-\frac{1}{2}} \frac{\partial}{\partial\nu^+} G(P) \quad ,$$

for every $P \in C$, where ν^- and ν^+ denote the interior and exterior normals of C, respectively.

If C is a sphere of radius R around Q, we have

(3.3.30) $$G(P,Q) = \frac{1}{4\pi}\left(\frac{1}{r(P,Q)} - \frac{1}{R}\right) \quad , \quad G(P) = \frac{1}{4\pi}\left(1 - \frac{R}{r(P,Q)}\right) \quad ;$$

hence

(3.3.31) $$C(Q)^{-\frac{1}{2}} \frac{\partial}{\partial\nu^-} G(P,Q) = \frac{1}{4\pi R^{3/2}} = C(\infty)^{-\frac{1}{2}} \frac{\partial}{\partial\nu^+} G(P) \quad .$$

Thus, the sphere is a stationary surface with respect to the functional π. We want now to show that it is the only possible one.

Suppose, indeed, there were another surface C_1 with this property. We take C_o to be a sphere inside C_1 with sufficiently small radius and we consider the level surfaces $C_t (0 \leq t \leq 1)$ of the harmonic function V(P) which is zero on C_o and 1 on C_1. Let

(3.3.32) $$\pi_t = \pi(C_t) = C_t(Q)\cdot C_t(\infty) \quad .$$

We have

(3.3.33) $$\pi_t' = C_t(\infty)C_t'(Q) + C_t'(\infty)C_t(Q) \quad ,$$

$$\Pi_t'' = C_t(\infty)C_t''(Q) + 2C_t'(\infty)C_t'(Q) + C_t''(\infty)C_t(Q) \quad . \tag{3.3.34}$$

Whenever Π_t' vanishes, we can write

$$\Pi_t'' = \Pi_t\left[\frac{C_t''(Q)}{C_t(Q)} + \frac{C_t''(\infty)}{C_t(\infty)} - 2\left(\frac{C_t'(\infty)}{C_t(\infty)}\right)^2\right] \quad . \tag{3.3.35}$$

In view of the convex dependence of $C_t(Q)$ on t and by (3.26), we find

$$\Pi_t'' \geqslant 0 \quad , \quad \text{if} \quad \Pi_t' = 0 \quad . \tag{3.3.36}$$

We have to consider the case of equality in (3.36). Clearly, this is only possible if $C_t''(Q) = 0$, and by virtue of (3.7) this is only possible if

$$\frac{\partial}{\partial t} G_t(P,Q) = \text{const.}, \quad \text{for } P \text{ inside } C_t \; . \tag{3.3.37}$$

By (3.3), this is only the case if we have on C_t the equality

$$\frac{\partial}{\partial \nu} G_t(P,Q) = \alpha \frac{\partial V(P)}{\partial \nu} \quad , \quad P \in C_t \quad , \tag{3.3.38}$$

with a suitable constant factor α. The surface C_t is an analytic surface, and by the Cauchy-Kowalewski uniqueness theorem for solutions of partial differential equations with given initial data, we obtain

$$G_t(P,Q) = \alpha[V(P) - t] \quad . \tag{3.3.39}$$

The sphere C_o appears as a level surface of $G_t(P,Q)$ and hence, obviously, all level surfaces C_t are spheres around the point Q. Hence the final surface C_1 will be a sphere around Q and we have the original type of stationary surface with respect to Π.

If C_1 is not a sphere with Q as center, we have the inequality

$$\Pi_t'' > 0 \quad , \quad \text{if} \quad \Pi_t' = 0 \quad . \tag{3.3.40}$$

This result shows that Π_t' can only cross from negative values to positive values as t increases. Since $\Pi_o' = 0$, we recognize that Π_t' remains positive for $t > 0$, and $\Pi_1' = 0$ is impossible. Thus, there cannot be a stationary surface other than a sphere around Q.

Since Π_t increased when we passed from C_o to C_1, we see that Π has its minimum value for this sphere and

$$(3.3.41) \qquad C(\infty)C(Q) \geqslant 1 \quad ,$$

with equality holding only in the case of a sphere around Q.

We observe how the convexity of the capacity leads to the inequality (3.3.41), a uniqueness theorem for the functional equation (3.29) and the monotonicity of Π_t at the same time.

4. The two-dimensional case.

All methods applied in the preceding sections can be carried over without any change to the case of a partial differential equation (2.8.1) in two independent variables. We devote to this case a special section only for the reason that some formulas simplify considerably and lend themselves to interesting applications.

Let again

$$(3.4.1) \qquad H(Q,R) = -\int_C \frac{\partial G(P,Q)}{\partial \nu_P} \frac{\partial G(P,R)}{\partial \nu_P} (S\cdot\nu)ds_P$$

denote the solution of the original differential equation which is, up to the factor ε, the first variation of the Green's function for the displacement vector field S_i $(i=1,2)$. If S_i has the normal direction on the boundary curve C of the plane domain D and if $\kappa(s)$ denotes the curvature of C at the point $P(s)$, we have in analogy to (2.16)

$$(3.4.2) \qquad \delta^2 G(Q,R) = -2\varepsilon^2 Q_o[H(P,Q),H(P,R)]$$
$$- \varepsilon^2 \int_C \frac{\partial G(P,Q)}{\partial \nu_P} \frac{\partial G(P,R)}{\partial \nu_P} \kappa(s)(S\cdot\nu)^2 ds \quad .$$

We want to extend this formula to the case when S_i is not normal to C; it is at this stage that the two-dimensional theory becomes simpler, in view

of the simpler differential geometry. We decompose the displacement vector S_i into normal and tangential components N_i and T_i,

$$S_i = N_i + T_i = (S\cdot\nu)\nu_i + [S_i - (S\cdot\nu)\nu_i] \quad , \tag{3.4.3}$$

and we let N and T be the projections of the vector S_i on the normal and tangential vectors, respectively. Starting again from the condition

$$x_i(s) + \varepsilon S_i(x_\ell) = x_i(s^*) + \varepsilon\lambda(s^*)\nu_i(s^*) \quad , \tag{3.4.4}$$

we derive by Frenet's formulas the relation

$$\varepsilon(N_i + T_i) = \dot{x}_i(s)\left\{(s^* - s) - \varepsilon\lambda(s^*)(s^* - s)\right\} + \nu_i(s)\left\{\frac{\kappa}{2}(s^* - s)^2 + \varepsilon\lambda(s^*)\right\} + o(\varepsilon^2) \quad . \tag{3.4.5}$$

Thus, we find

$$s^* - s = \varepsilon T + o(\varepsilon) \quad , \tag{3.4.6}$$

$$\lambda(s^*) = N - \varepsilon\frac{\kappa}{2}T^2 + o(\varepsilon^2) \quad , \tag{3.4.7}$$

and finally

$$\lambda(s) = N - \varepsilon\left[T\frac{dN}{ds} + \frac{\kappa}{2}T^2\right] + o(\varepsilon^2) \quad . \tag{3.4.8}$$

This is the equivalent normal displacement of C under the general deformation (4.3).

Because of the correction in the normal shift, the first order variation leads to a correction term in the second variation and we find

$$\varepsilon^2 G(Q,R) = -2\varepsilon^2 Q_0[H(P,Q),H(P,R)] - \varepsilon^2\int_C \frac{\partial G(P,Q)}{\partial\nu_P}\frac{\partial G(P,R)}{\partial\nu_P}\left[\kappa N^2 - 2T\frac{dN}{ds} - \kappa T^2\right]ds \quad . \tag{3.4.9}$$

This formula gives the second variation of the Green's function under an arbitrary displacement with a vector field S_i.

Formula (4.9) enables us to study the variation of the Green's function with the parameter of a one-parameter family of curves. Let, for example, C_t be a family of closed curves given by the complex parametric representation

$$z = f(\sigma;t) \quad , \quad 0 \leq t \leq 1, \quad 0 \leq \sigma \leq 1 \quad , \tag{3.4.10}$$

which expresses for fixed t the points of the curve C_t in dependence on the parameter σ. We assume the function f to be periodic in σ with period 1 and to have two continuous derivatives in each variable.

Each curve C_t defines a finite domain D_t and a corresponding Green's function $G(z,\zeta;t)$. We consider the curve C_{t_o} and the shift

$$f(\sigma;t) - f(\sigma;t_o) = f_t(\sigma;t_o)(t-t_o) + \frac{1}{2} f_{tt}(\sigma;t_o)(t-t_o)^2 + \ldots , \tag{3.4.11}$$

which carries points from C_{t_o} into points of C_t. We may express the unit tangent vector at the point $z(\sigma;t_o)$ in the form

$$e^{i\alpha} = f_\sigma(\sigma;t_o)/|f_\sigma(\sigma;t_o)| \tag{3.4.12}$$

and decompose f_t into tangential and normal components,

$$f_t(\sigma;t_o)(t-t_o) = T e^{i\alpha} + N i e^{i\alpha} \quad , \tag{3.4.12'}$$

with

$$T = \operatorname{Re}\left\{f_t(\sigma;t_o)e^{-i\alpha}\right\}(t-t_o) \; , \; N = \operatorname{Im}\left\{f_t(\sigma;t_o)e^{-i\alpha}\right\}(t-t_o) \; . \tag{3.4.12''}$$

Using (3.4.9) and observing the additional correction from the first variation due to the term $\frac{1}{2} f_{tt}(\sigma;t_o)(t-t_o)^2$ in the displacement, we find

$$\frac{\partial^2 G(z,\zeta;t)}{\partial t^2} = -2Q_o[H(w,z;t),H(w,\zeta;t)] - \int_C \frac{\partial G(w,z;t)}{\partial \nu_w} \frac{\partial G(w,\zeta;t)}{\partial \nu_w} [\kappa|f_t(\sigma;t)|^2 - 2\operatorname{Re}\left\{f_t e^{-i\alpha}\right\}\operatorname{Im}\left\{f_{\sigma t} e^{-i\alpha}\right\}/|f_\sigma| + \operatorname{Im}\left\{f_{tt}(\sigma;t)e^{-i\alpha}\right\}]ds_\sigma \quad . \tag{3.4.13}$$

This formula becomes particularly interesting in the case of convex curves, for which the curvature κ is always positive. Let C_o and C_1 be any two convex curves enclosing the origin. We suppose them described analytically by the values $p_o(\varphi)$, $p_1(\varphi)$ of the supporting functions in dependence on the angle of inclination φ of the normal. We introduce then the family of convex curves C_t by defining their supporting functions as follows:

(3.4.14) $$p_t(\varphi) = (1-t)p_o(\varphi) + tp_1(\varphi) \quad , \quad 0 \leq t \leq 1 \quad .$$

In order to describe this family in the complex form (4.10), we observe that a tangent to C_t whose normal has the angle of inclination φ is given by Hesse's normal form

(3.4.15) $$x \cos \varphi + y \sin \varphi = p_t(\varphi)$$

and that C_t is the envelope of this family of straight lines. Hence, the points of C_t are found by eliminating φ from (4.15) and

(3.4.16) $$-x \sin \varphi + y \cos \varphi = p_t'(\varphi) \quad .$$

Thus, we find easily

(3.4.17) $$f(\varphi;t) = e^{i\varphi}[p_t(\varphi) + ip_t'(\varphi)]$$

$$= (1-t)f(\varphi;0) + tf(\varphi;1) \quad .$$

We calculate

(3.4.18) $$f_\varphi = ie^{i\varphi}[p_t(\varphi) + p_t''(\varphi)]$$

and verify that $e^{i\alpha} = -ie^{i\varphi}$ and

(3.4.19) $$\operatorname{Im}\left\{f_{\varphi t}\, e^{-i\alpha}\right\} = 0 \quad .$$

Moreover, we have clearly

(3.4.20) $$f_{tt}(\varphi;t) \equiv 0 \quad .$$

Thus, (4.13) simplifies to

$$(3.4.21)\qquad \frac{\partial^2 G(z,\zeta;t)}{\partial t^2} = -2Q_o[H(w,z;t),H(w,\zeta;t)]$$
$$-\int_C \frac{\partial G(w,z;t)}{\partial \nu_w}\frac{\partial G(w,\zeta;t)}{\partial \nu_w} K|f(\varphi;1)-f(\varphi;0)|^2 ds_\varphi \quad .$$

Formula (4.21) is very useful in the case that the coefficient $p(x_i)$ of the differential equation is non-negative. In this case, we can again assert that $Q_o[u]$ is non-negative and that $\frac{\partial G}{\partial \nu}$ is non-negative. Let $S(z,\zeta)$ be a fundamental singularity of the differential equation considered; then

$$(3.4.22)\qquad h(z,\zeta;t) = G(z,\zeta;t) - S(z,\zeta)$$

is a regular solution of the differential equation in D_t. Reasoning as in Section 3, we can assert that the expression

$$(3.4.23)\qquad A_t = \sum_{\rho,\sigma=1}^{N} h(z_\rho,z_\sigma;t)\lambda_\rho\lambda_\sigma$$

is a concave function of the parameter t.

This general result contains numerous special cases and leads to various inequalities. It implies, in particular, that the capacity of a convex family (4.14) at a given point with respect to Laplace's equation is a convex function of the parameter t of the family. We do not enter here into a discussion of the various results and their possible generalizations.

5. Singular variations of the second order.

In Section 9 of Chapter II, we derived from the general formula for first order variations of the Green's function of a domain D in the complex z-plane a particular result for the deformation

$$(3.5.1)\qquad z^* = z + \frac{e^{i\alpha}\rho^2}{z - z_o} \quad .$$

This formula has in the case of Laplace's equation the simple form

$$G^*(\zeta^*, \eta^*) - G(\zeta, \eta) = \mathrm{Re}\left\{8\pi\rho^2 e^{i\alpha} \frac{\partial G(z_0, \zeta)}{\partial z_0} \frac{\partial G(z_0, \eta)}{\partial z_0}\right\} + o(\rho^2) \quad . \tag{3.5.2}$$

Since this result has been derived from the formula (2.8.7), it is valid for the most general plane domains and can be applied to extremum problems for the Green's function in order to characterize the extremal domain. One obtains in many cases, from the extremum requirement and the variational formula, necessary conditions for the extremum domain in the form of a differential equation for its Green's function. One can then show easily that the required extremum domain has a piecewise analytic boundary curve C; one can even find in many cases domains D which satisfy all necessary conditions imposed by the study of the first order variation. But the question arises to show that there exists only one unique domain satisfying the necessary conditions. It is natural to study the theory of the second variation in order to obtain such uniqueness results.

We want to derive in this section a theory of the second variation of the harmonic Green's function under interior variations of the type (5.1). Since we cannot use the principle of superposition of variations when studying second order terms, we shall start immediately with a variation

$$z^* = z + \varepsilon \sum_{\nu=1}^{N} \frac{\lambda_\nu}{z - z_\nu} \quad , \qquad |\lambda_\nu| \leq 1 \ , \ \nu = 1, \ldots, N \quad . \tag{3.5.3}$$

We suppose that no z_ν lies on the boundary C of the original domain D; if ε is small enough, the mapping (5.3) will carry C into a curve C^* in a one-to-one manner, and let D^* be the interior of C^*. We denote by $G^*(\zeta, \eta)$ the Green's function of the varied domain.

We assume that C is an analytic curve; by our preceding remarks it is clear that this case will be the most important one in the applications. Moreover, because of the continuous dependence of the Green's function on the

domain, our final result can be extended to the most general domains through approximation by analytically bounded ones. We shall follow the same method used in Section 2 for the general case, but shall utilize fully the advantages of complex notation.

Since D and D^* have analytic boundaries, we can choose ε so small that both Green's functions G^* and G are analytic in the closure of $D + D^*$. Writing $z^* = z + \varepsilon Z(z)$, we have by Taylor's theorem

$$(3.5.4) \quad 0 = G^*(\zeta + \varepsilon Z(\zeta), \eta) = G^*(\zeta, \eta) + 2 \operatorname{Re}\left\{\varepsilon Z(\zeta) \frac{\partial G^*(\zeta, \eta)}{\partial \zeta} + \frac{\varepsilon^2}{2} Z(\zeta)^2 \frac{\partial^2 G^*(\zeta, \eta)}{\partial \zeta^2}\right\} + O(\varepsilon^3)$$

for $\zeta \in C$. Hence, the function

$$(3.5.5) \qquad \Lambda(\zeta, \eta) = G^*(\zeta, \eta) - G(\zeta, \eta)$$

is regular harmonic in the closure of D and has for $\zeta \in C$ the boundary values

$$(3.5.4') \qquad -2 \operatorname{Re}\left\{\varepsilon Z(\zeta) \frac{\partial G^*(\zeta, \eta)}{\partial \zeta} + \frac{\varepsilon^2}{2} Z(\zeta)^2 \frac{\partial^2 G^*(\zeta, \eta)}{\partial \zeta^2}\right\} + O(\varepsilon^3) .$$

Using the Green's function $G(\zeta, \eta)$ of D and the identity

$$(3.5.6) \qquad \frac{\partial G(z, \zeta)}{\partial n_z} ds_z = -\frac{2}{i} \frac{\partial G(z, \zeta)}{\partial z} dz \quad ,$$

we obtain

$$(3.5.7) \quad \Lambda(\zeta, \eta) = \operatorname{Re}\left\{\frac{4}{i} \int_C \left[\varepsilon Z(z) \frac{\partial G^*(z, \eta)}{\partial z} + \frac{\varepsilon^2}{2} Z(z)^2 \frac{\partial^2 G^*(z, \eta)}{\partial z^2}\right] \cdot \frac{\partial G(z, \zeta)}{\partial z} dz\right\} + O(\varepsilon^3) \quad .$$

Put now

$$(3.5.7') \qquad \Lambda(\zeta, \eta) = \varepsilon \Lambda_1(\zeta, \eta) + \frac{\varepsilon^2}{2} \Lambda_2(\zeta, \eta) + O(\varepsilon^3) \quad .$$

We find at once

$$(3.5.8) \quad \Lambda_1(\zeta, \eta) = \operatorname{Re}\left\{\frac{4}{i} \int_C Z(z) \frac{\partial G(z, \eta)}{\partial z} \frac{\partial G(z, \zeta)}{\partial z} dz\right\} \quad ,$$

$$(3.5.9) \quad \Lambda_2(\zeta,\eta) = \mathrm{Re}\left\{\frac{4}{i}\int_C [2Z(z)\frac{\partial\Lambda_1(z,\eta)}{\partial z} + Z(z)^2\frac{\partial^2 G(z,\eta)}{\partial z^2}]\frac{\partial G(z,\zeta)}{\partial z}dz\right\}.$$

We can evaluate the integrals by means of the residue theorem. We find, first,

$$(3.5.10) \quad \Lambda_1(\zeta,\eta) = \mathrm{Re}\left\{8\pi\sum_{\nu=1}^{N}\lambda_\nu\frac{\partial G(z_\nu,\eta)}{\partial z_\nu}\frac{\partial G(z_\nu,\zeta)}{\partial z_\nu} - 2Z(\eta)\frac{\partial G(\eta,\zeta)}{\partial\eta} - 2Z(\zeta)\frac{\partial G(\eta,\zeta)}{\partial\zeta}\right\} .$$

This result is equivalent to (5.2) and could have been obtained from this formula by the principle of superposition of first order variations.

In order to calculate the second variation term $\Lambda_2(\zeta,\eta)$, it is useful to introduce the two kernels

$$(3.5.11) \qquad K(\zeta,\overline{\eta}) = -4\frac{\partial^2 G(\zeta,\eta)}{\partial\zeta\,\partial\overline{\eta}}$$

and

$$(3.5.12) \qquad L(\zeta,\eta) = -4\frac{\partial^2 G(\zeta,\eta)}{\partial\zeta\,\partial\eta} = \frac{1}{\pi(\zeta-\eta)^2} - \ell(\zeta,\eta) \quad ,$$

which play an important role in the theory of conformal mapping and orthogonal function systems in the domain D [1]. The kernels $K(\zeta,\overline{\eta})$ and $\ell(\zeta,\eta)$ are regular analytic functions of their arguments throughout the domain D.

In order to simplify the result, let us suppose that the parameters λ_ν in (5.3) have been chosen in such a way that

$$(3.5.13) \qquad Z(\zeta) = Z(\eta) = 0 \quad ,$$

i.e., so that the argument points considered are fixed under the variation. With these notations and assumptions, we calculate easily

$$(3.5.14) \quad \Lambda_2(\zeta,\eta) = \mathrm{Re}\left\{16\pi^2\sum_{\mu,\nu=1}^{N}\lambda_\nu\lambda_\mu\frac{\partial G(z_\nu,\eta)}{\partial z_\nu}\frac{\partial G(z_\mu,\zeta)}{\partial z_\mu}\ell(z_\mu,z_\nu) - 16\pi^2\sum_{\mu,\nu=1}^{N}\overline{\lambda}_\nu\lambda_\mu\frac{\partial G(z_\nu,\eta)}{\partial\overline{z}_\nu}\frac{\partial G(z_\mu,\zeta)}{\partial z_\mu}K(z_\mu,\overline{z}_\nu) + 8\pi\sum_{\mu,\nu=1}^{N}\lambda_\nu\lambda_\mu\frac{1}{(z_\mu-z_\nu)^2}\left(\frac{\partial G(z_\mu,\eta)}{\partial z_\mu}-\frac{\partial G(z_\nu,\eta)}{\partial z_\nu}\right)\left(\frac{\partial G(z_\mu,\zeta)}{\partial z_\mu}-\frac{\partial G(z_\nu,\zeta)}{\partial z_\nu}\right)\right\}$$

In this formula for the second interior variation of the Green's function, certain bilinear forms appear which are well known in the general theory of conformal mapping. The forms

$$(3.5.15) \qquad \tilde{L} = \sum_{\mu,\nu=1}^{N} \ell(z_\mu, z_\nu) x_\mu x_\nu \ , \quad \tilde{K} = \sum_{\mu,\nu=1}^{N} K(z_\mu, \bar{z}_\nu) x_\mu \bar{x}_\nu$$

satisfy, for example, the inequality

$$(3.5.15') \qquad |\tilde{L}| \leq \tilde{K} \quad ,$$

a result which leads to various convexity statements with respect to functionals connected with the Green's function.

We do not intend to make applications of the above formulas in the present paper. Our main purpose in presenting the results is to show the uniform character of the various methods applied in the variational calculus.

6. <u>The comparison method</u>.

In this section, we want to give an alternate method for the derivation of second order variational formulas and apply this method to the Neumann's function.

We deal again with a three-dimensional domain D and the partial differential equation (2.1.1). Let D^* be another domain and suppose that the intersection $D_2 = D \cdot D^*$ is a domain. Let $N(P,Q)$ and $N^*(P,Q)$ denote the Neumann's functions of the domains D and D^*, respectively. We continue the definition of these functions beyond their domain by putting them equal to zero if an argument point lies outside of this domain. We suppose that both domains are smoothly bounded. Let Q and R be two points in the intersection D_2 and let $Q_D[u,v]$ denote the bilinear Dirichlet integral for the differential equation (2.1.1) extended over the domain D. It is then easy to verify the identity

$$(3.6.1) \quad N^*(Q,R) - N(Q,R) = Q_{D+D*}[N^*(P,Q) - N(P,Q), N^*(P,R) - N(P,R)] \quad ,$$

which is a consequence of Green's theorem. Exactly the same identity, but with opposite signs, is also true if the Neumann's functions are replaced by the corresponding Green's functions.

Let us suppose now that the domains D and D^* are very near to each other and that their boundary surfaces lie in an ε neighborhood of each other. In this case, in each closed subdomain of the intersection D_2 we have a uniform estimate

$$N^*(P,Q) - N(P,Q) = O(\varepsilon) \quad . \tag{3.6.2}$$

Hence, the contribution of D_2 to the above Dirichlet integral is only of the order ε^2. The first order terms of the Dirichlet integral are due to the difference shells and lead to the equation

$$N^*(Q,R) - N(Q,R) = \iint_C [\nabla N(P,Q) \cdot \nabla N(P,R) + p(P)N(P,Q)N(P,R)]\,\delta\nu_P d\sigma_P + o(\varepsilon) , \tag{3.6.3}$$

which is Hadamard's variational formula (2.10.15).

If we want to evaluate the identity (6.1) to a higher order of precision in ε, we have to utilize the following result. Let C be a sufficiently smooth surface and Σ a thin shell over C with variable width $\delta\nu$. Let f(P) be a twice continuously differentiable function in the closure of Σ; then we have the estimate

$$\iiint_\Sigma f(P)d\tau = \iint_C f(P)\delta\nu\, d\sigma + \iint_C [\tfrac{1}{2}\frac{\partial f}{\partial\nu} - \tfrac{1}{2}(\frac{1}{\rho_1} + \frac{1}{\rho_2})f]\,\delta\nu^2 d\sigma + o(\delta\nu^2) , \tag{3.6.4}$$

where ρ_1 and ρ_2 are again the principal radii of curvature at the point P of integration. Applying this general formula to estimate (6.1), we find easily

$$\delta^2 N(Q,R) = 2Q_D[\delta N(P,Q), \delta N(P,R)] + \iint_C \Big\{\frac{\partial}{\partial\nu}[\nabla N(P,Q) \cdot \nabla N(P,R)] \tag{3.6.5}$$

$$+ \frac{\partial p}{\partial\nu} N(P,Q)N(P,R) - (\frac{1}{\rho_1} + \frac{1}{\rho_2})[\nabla N(P,Q) \cdot \nabla N(P,R) + pN(P,Q)N(P,R)]\Big\}\delta\nu_P^2 d\sigma_P .$$

It is worth while to consider in more detail the term $\frac{\partial}{\partial \nu}(\nabla N(P,Q) \cdot \nabla N(P,R))$. Let $\mathfrak{r}(u,v)$ denote the vector varying on the surface C and put

$$\nabla N(P,Q) = \alpha \mathfrak{r}_u + \beta \mathfrak{r}_v \;, \quad \nabla N(P,R) = \gamma \mathfrak{r}_u + \delta \mathfrak{r}_v \quad . \tag{3.6.6}$$

This representation is possible, since ∇N has no normal component on C. We can write

$$\frac{\partial}{\partial \nu}[\nabla N(P,Q) \cdot \nabla N(P,R)] = \sum_{i,k=1}^{3} \frac{\partial^2 N(P,Q)}{\partial x_i \, \partial x_k} \frac{\partial N(P,R)}{\partial x_k} \nu_i + \sum_{i,k=1}^{3} \frac{\partial^2 N(P,R)}{\partial x_i \, \partial x_k} \frac{\partial N(P,Q)}{\partial x_k} \nu_i \quad , \tag{3.6.7}$$

where $\underline{\nu}$ is the normal vector to C,

$$\underline{\nu} = \frac{1}{\mathcal{H}} (\mathfrak{r}_u \times \mathfrak{r}_v) \quad , \quad \mathcal{H} = |r_u \times r_v| \quad . \tag{3.6.8}$$

We start now from the identity

$$\sum_{i=1}^{3} \frac{\partial N}{\partial x_i} \nu_i = 0 \quad , \tag{3.6.9}$$

and we obtain by differentiation

$$\sum_{i,k=1}^{3} \frac{\partial^2 N}{\partial x_i \, \partial x_k} \frac{\partial x_k}{\partial u} \nu_i = - \sum_{i=1}^{3} \frac{\partial N}{\partial x_i} \frac{\partial \nu_i}{\partial u} \tag{3.6.10}$$

and an analogous equation with v replacing u. Thus, using (6.6), we can write

$$\sum_{i,k=1}^{3} \frac{\partial^2 N(P,Q)}{\partial x_i \, \partial x_k} \frac{\partial N(P,R)}{\partial x_k} \nu_i = - \sum_{i=1}^{3} \frac{\partial N(P,Q)}{\partial x_i} \left(\gamma \frac{\partial \nu_i}{\partial u} + \delta \frac{\partial \nu_i}{\partial v}\right) \tag{3.6.11}$$

$$= -(\alpha \mathfrak{r}_u + \beta \mathfrak{r}_v) \cdot (\gamma \underline{\nu}_u + \delta \underline{\nu}_v) \quad .$$

It is well known that the relations

$$\mathfrak{r}_u \cdot \nu_u = -\mathcal{L} \;, \quad \mathfrak{r}_v \cdot \nu_u = \mathfrak{r}_u \cdot \nu_v = -\mathcal{M} \;. \quad \mathfrak{r}_v \cdot \nu_v = -\mathcal{N} \tag{3.6.12}$$

hold, when $\mathcal{L}, \mathcal{M}, \mathcal{N}$ denote again the coefficients of the second fundamental form. Thus, we can put

$$(3.6.13)\qquad \frac{\partial}{\partial \nu}[\nabla N(P,Q)\cdot\nabla N(P,R)] = 2[\alpha\gamma\mathcal{L} + \mathcal{M}(\alpha\delta + \beta\gamma) + \mathcal{N}\beta\delta]$$

$$= 2\nabla N(P,Q)\cdot\nabla N(P,R)\,\frac{\alpha\gamma\mathcal{L} + (\alpha\delta + \beta\gamma)\mathcal{M} + \beta\delta\mathcal{N}}{\alpha\gamma E + (\alpha\delta + \beta\gamma)F + \beta\delta G}\;.$$

Thus, the normal derivative of the scalar product between two Neumann's functions equals the scalar product itself, multiplied by the ratio of the second and the first bilinear fundamental forms. It is remarkable that his expression involves only first order derivatives of the Neumann's functions.

We want to formulate our result for the particular case of Laplace's equation. We find

$$(3.6.14)\quad \delta^2 N(Q,R) = 2Q_D[\delta N(P,Q),\delta N(P,R)]$$

$$+ \iint_C \nabla N(P,Q)\cdot\nabla N(P,R)\Big[2\mathcal{R}\Big\{\nabla N(P,Q),\nabla N(P,R)\Big\} - \Big(\frac{1}{\rho_1} + \frac{1}{\rho_2}\Big)\Big]\delta\nu_P^2\,d\sigma_P\;.$$

Here, $\mathcal{R}(\underline{v}_1,\underline{v}_2)$ denotes the ratio of the second and the first fundamental forms for any two given tangential vectors $\underline{v}_1$ and $\underline{v}_2$. This result can only be applied in the case that the domain D considered is the exterior of a closed surface C, since only in this case can the existence of a Neumann's function in the proper sense oe assumed.

We proceeded in this section in a rather formal way, without specifying the exact assumptions under which the above estimates can be justified. It is, however, sufficient to point out such a formal method which works at least in the case of analytic surfaces C; for, we know by our general theory of variation fields that a formula of the type (6.5) for the second variation of the Neumann's function must exist and can be obtained by integration by parts of the interior variational formula. The method of the present section serves only to give a short cut to the final result without too much labor.

Next, let $\varphi_1(P)$ denote the velocity potential for an incompressible and irrotational fluid flow around a surface C which has at infinity unit velocity in the direction of the x_1-axis. This function has been discussed in Section 12 of Chapter II, and its first order variation was determined there. Let α_{11} denote the corresponding coefficient defined by (2.12.17), which is closely related to the virtual mass of C with respect to the direction x_1. Let D be the exterior of C and let C^* be another closed surface with exterior D^* such that $D_2 = D \cdot D^*$ is not empty. If α^*_{11} and φ^*_1 denote the corresponding quantities for the new domain, we find

$$(3.6.15) \qquad \alpha^*_{11} - \alpha_{11} = \frac{1}{4\pi} Q_{D+D^*}[\varphi^*_1 - \varphi_1] \qquad ,$$

where Q is the Dirichlet integral for Laplace's equation. This formula is analogous to (6.1) and can be derived also by application of Green's identity. It is also to be understood here that φ_1 and φ^*_1 are defined to be zero outside of their domains of definition.

Evaluating (6.15) the same way as we evaluated (6.1), we find

$$(3.6.16) \qquad \delta\alpha_{11} = \frac{1}{4\pi} \iint_C (\nabla \varphi_1)^2 \delta\nu \, d\sigma \qquad ,$$

which is already given in (2.12.21), and

$$(3.6.17) \quad \delta^2\alpha_{11} = \frac{1}{2\pi} Q_D[\delta\varphi_1] + \frac{1}{4\pi} \iint_C [\frac{\partial}{\partial\nu}(\nabla\varphi_1)^2 - (\frac{1}{\rho_1} + \frac{1}{\rho_2})(\nabla\varphi_1)^2] \delta\nu^2 d\sigma .$$

Finally, we want to give the expression for the second variation of α_{11} in the case of an axially symmetric surface C, under the assumption that the variation $\delta\nu$ is also performed in an axially symmetric way. Let x,y be the coordinates in the meridian plane and let $y = 0$ be the axis of symmetry. We denote the meridian curve of C in this plane by c. If κ is the curvature of c at a point (x,y), we have

$$(3.6.18) \qquad \frac{1}{\rho_1} + \frac{1}{\rho_2} = \left(K + \frac{1}{y\sqrt{1+y'^2}} \right)$$

for the corresponding points P on C, since the domain D considered lies outside of C. Using the fact that because of the axial symmetry the velocity vector $\nabla \varphi_1$ lies always in the meridian plane, we calculate easily by the method of this section that

$$(3.6.19) \qquad \frac{\partial}{\partial \nu} (\nabla \varphi_1)^2 = -2 K(\nabla \varphi_1)^2 \quad .$$

Hence, we have

$$(3.6.20) \qquad \delta^2 \alpha_{11} = \frac{1}{2\pi} Q_D[\delta \varphi_1] + \frac{1}{2} \int_C (\nabla \varphi_1)^2 \left(\frac{1}{\sqrt{1+y'^2}} - Ky\right) \delta \nu^2 \, ds \; .$$

Consider finally the case where C is a rotationally symmetric vortex sheet. In this case, $(\nabla \varphi_1)^2$ will have equal values on the upper and the lower side of the sheet. The curvatures of C will have to be taken with opposite signs on the upper and lower side, however, so that in this case formula (6.20) reduces to

$$(3.6.21) \qquad \delta^2 \alpha_{11} = \frac{1}{2\pi} Q_D[\delta \varphi_1] > 0 \quad .$$

Since, by its very definition, the vortex sheet satisfies the condition $\delta \alpha_{11} = 0$, we recognize that it represents a local minimum for the coefficient α_{11}.

We shall give another formula (4.3.1) for the second variation of the virtual mass of a body of rotation. The stream function Ψ will enter instead of the velocity potential φ_1 used in (6.20). Both formulas are, of course, equivalent, but (4.3.1) is better adapted to the uniqueness question treated in Section 3 of Chapter IV.

CHAPTER IV

AXIALLY SYMMETRIC VORTEX SHEETS

1. Variational formulation of the problem.

As an application of the methods of Chapter II and as a first step in the solution of the problem of 3 -dimensional vortex sheets discussed in Chapter I, we formulate and solve a minimum problem which gives a construction for vortex sheets in axially symmetric steady irrotational flow of an incompressible fluid.

Let B denote a compact connected set in the half-plane $y \geqq 0$ which intersects the x -axis, let b denote a point in the infinite exterior component of B relative to $y > 0$ and let W be a continuum in the half-plane $y \geqq 0$ which, together with the x-axis, joins b to B in the sense that the x -axis and B+W+b form a connected set. We denote by D the infinite region complementary to B+W+b in the half-plane $y \geqq 0$ and we consider an axially symmetric flow parallel to the x -axis in the 3 -dimensional region obtained by revolving D about the x -axis. This flow is governed by a velocity potential φ and a stream function ψ which are functions of x and y only and satisfy the generalized Cauchy-Riemann equations

$$(4.1.1) \qquad \varphi_x = \psi_y/y \quad , \quad \varphi_y = -\psi_x/y \quad .$$

in D and the boundary conditions

$$(4.1.2) \qquad \psi = \frac{\partial \varphi}{\partial \nu} = 0$$

on the curves C bounding D. We normalize ψ at infinity to have the behavior

$$(4.1.3) \qquad \psi = \frac{y^2}{2} - \frac{\alpha y^2}{r^3} + \ldots \quad , \quad r^2 = x^2 + y^2 \quad ,$$

and we call α the virtual mass of the flow.

We consider B and b to be fixed, and we attempt to choose the variable continuum W so that

(4.1.4) $\qquad \alpha =$ minimum .

We shall prove that a unique extremal continuum W for (1.4) exists, is an analytic curve connecting b with B or with the x-axis, and generates a surface of revolution about the x-axis characterized as a vortex sheet for the flow Ψ.

As a preliminary, we point out that only a bounded class of continua W need be considered for the extremal problem (4.1.4). For if W rises to a height h above the x-axis, we can show by symmetrization of B+W+b in the y-axis, which decreases α, that the virtual mass α of B+W+b is larger than the virtual mass $\alpha(h)$ of the flow past a disc of radius h perpendicular to the x-axis [7, 9]. Since $\alpha(h) \to \infty$ as $h \to \infty$, there is no loss of generality in requiring that B+W+b lie in a suitable high strip $0 \leq y \leq H$. Suppose now that W is a curve lying within a very large strip $|x| \leq I$, but in no smaller vertical strip. Let Ψ^* be the stream function of the flow of the type (1.3) in the exterior of the rectangle $|x| \leq I$, $0 \leq y \leq H$ in the upper half-plane; and let Ψ^{**} be a solution of (1.1), defined in one of the two rectangles obtained from $|x| \leq I$, $0 \leq y \leq H$ by drawing a vertical line through b, which has a suitable large positive value on this line and which vanishes on the remainder of the boundary of the rectangle. If W touches the lines $|x| = I$ at a point z_o in the half-plane $y > 0$ near z_o let Ψ^+ denote the branch of Ψ which is defined outside the rectangle $|x| \leq I$, $0 \leq y \leq H$, and let Ψ^- be the branch of Ψ defined on the other side of W. Then $\Psi^* < \Psi^+$ and $\Psi^- < \Psi^{**}$ near z_o, by the maximum principle for (1.1). Hence at z_o, for large I,

$$(4.1.5) \qquad \frac{\partial \Psi^+}{\partial \nu} > \frac{\partial \Psi^*}{\partial \nu} > \frac{\partial \Psi^{**}}{\partial \nu} > \frac{\partial \Psi^-}{\partial \nu} \quad ,$$

since as $I \to \infty$ we have $\psi^{**} \to 0$ and $\psi^{*} \not\to 0$ near z_o. But Hadamard's formula

$$(4.1.6) \qquad \delta\alpha = \frac{1}{2}\int \left(\frac{\partial \psi}{\partial \nu}\right)^2 \frac{\delta \nu \, ds}{y}$$

can be applied to show that because of this inequality, a shift of the curve W inwards, defined by bringing the vertical lines $x = \pm I$ closer together and by replacing arcs of W beyond these lines by vertical segments, will diminish α. Indeed, α can be diminished thus until I is so small that

$$(4.1.7) \qquad \frac{\partial \psi^*}{\partial \nu} = \frac{\partial \psi^{**}}{\partial \nu} \quad ,$$

and hence we obtain bounds on the height and width of competing continua W.

We shall find it more convenient to replace in D the stream function ψ by the positive solution

$$(4.1.8) \qquad u = \psi/y^{1/2}$$

of the self-adjoint partial differential equation

$$(4.1.9) \qquad \nabla^2 u = (3/4y^2)u \quad .$$

From a minimal sequence of domains D_n for (1.4) whose virtual mass coefficients α_n approach their greatest lower bound, we find that we can select a subsequence, again denoted by D_n, which converges in the sense of Carathéodory [3] to a limit domain D. The corresponding functions u_n form an equicontinuous family in each closed subdomain of D, whence by suitable extraction of a further subsequence we can achieve that u_n tend to a limit solution u of (1.9). Each u_n is positive in D_n and hence is subharmonic there, by (1.9). Thus if we let U_n be a harmonic function, defined in the region D_n^* obtained by excluding the exterior of a large circle R from D_n, which vanishes on the boundary of D_n and is equal on R to the maximum of u_n on R, then $u_n \leq U_n$ in D_n^*. As the domains D_n converge to D, it is known that the harmonic functions

U_n approach a harmonic limit U which vanishes on the boundary of D. Hence the limit solution u of (1.9) in D must vanish on the boundary C of D, since $0 \leq u \leq U$ in D^*.

We conclude that $\psi = uy^{1/2}$ is the stream function of a parallel flow (1.3) in D, and therefore the virtual mass α for D has the desired extremal property (1.4).

In the neighborhood of a point z_o of the extremal continuum W bounding D, chosen in the half-plane $y > 0$ and not in B+b, we make an admissable variation of the form

$$z^* = z + \frac{\varepsilon \omega}{z - t} \quad , \tag{4.1.10}$$

where ω is an infinitely differentiable function which is identically 1 in a neighborhood Ω of z_o and vanishes identically outside a slightly larger neighborhood. By Chapter II, formula (2.9.4), the variation of virtual mass α under such a shift is found to be

$$\delta\alpha = \mathrm{Re}\left\{ 2\varepsilon \iint_{D\Omega} u^2 \frac{\partial}{\partial z}\left[\frac{3}{4y^2(z-t)}\right] dxdy - 8\pi\varepsilon\,\delta(t)\left[\frac{\partial u}{\partial t}\right]^2 + \varepsilon Q_o(t) \right\} + o(|\varepsilon|^2) \quad , \tag{4.1.11}$$

where $Q_o(t)$ is analytic for t in Ω and where $\delta(t)$ is 1 for t in D and 0 for t exterior to D. Two applications of Green's theorem yield

$$\delta\alpha = \mathrm{Re}\left\{ \varepsilon \iint_{D\Omega} \frac{\bar{z} - \bar{t}}{z - t}\left[\frac{3u_z u_{\bar{z}}}{y^2} + \frac{9u^2}{16y^4} + \frac{3uu_z}{iy^3}\right] dxdy - 8\pi\varepsilon\,\delta(t) u_t^2 + \varepsilon Q(t) \right\} + o(|\varepsilon|^2) \quad , \tag{4.1.12}$$

where Q(t) is another analytic function of t in Ω.

From the extremal property (1.4) of α, we conclude that $\delta\alpha \geq 0$ for all sufficiently small complex ε. Hence, by the usual argument

$$8\pi\delta(t) u_t^2 = \iint_{D\Omega} \frac{\bar{z} - \bar{t}}{z - t}\left[\frac{3u_z u_{\bar{z}}}{y^2} + \frac{9u^2}{16y^4} + \frac{3uu_z}{iy^3}\right] dxdy + Q(t) \tag{4.1.13}$$

in Ω. We conclude from (1.13) that u_t^2 is continuous throughout Ω, if we define it to be zero in the exterior of D. The continuity of u_t^2 in the neighborhood of each point z_o of W is the variational condition imposed upon W by (1.4) which will lead us ultimately to deduce that revolving W about the x -axis generates a vortex sheet.

2. <u>Regularity of the solution</u>.

Since u_t^2 is continuous in Ω, we can estimate the integrand in (1.13) once again and prove that u_t^2 actually satisfies a Lipschitz condition there. Hence in the region $|u_t| > 0$ we can solve the differential equation

(4.2.1) $$u_t dt + u_{\bar{t}} d\bar{t} = 0$$

for the level curves of u. In particular, any arcs of the boundary $u = 0$ of D in the region $|u_t| > 0$ must be differentiable curves. If we pick Ω to lie in this region, we can apply (1.13) repeatedly to show that u_t^2 has continuous derivatives of several orders. Thus for a fixed determination of the square roots involved, we show using the proof of the reflection principle that

(4.2.2) $$u = \int \left\{ (u_z^2)^{1/2}\, dz + (u_{\bar{z}}^2)^{1/2}\, d\bar{z} \right\}$$

is an analytic solution of (1.9) in the region $|u_t| > 0$, and the boundary arcs $u = 0$ of D there must be analytic. Formula (2.2) gives an analytic continuation of u across these arcs according to which u transforms into its negative as we cross the boundary, and is therefore two-valued.

In order to see that there actually exist arcs of W in the region $|u_t| > 0$, it is sufficient to notice that every point z_o of W which can be touched by a small circle R lying in D is of this type. For if g is the Green's function of (1.9) in R with its infinity at the center of R, then for a small enough $\varepsilon > 0$, we find that in a neighborhood of z_o in R

(4.2.3) $$u \geqslant \varepsilon g \quad .$$

Since u and g both vanish at z_o, we can take normal derivatives there to obtain

(4.2.4) $$\frac{\partial u}{\partial \nu} \geq \varepsilon \frac{\partial g}{\partial \nu} ,$$

whence $|u_t| \geq \varepsilon |g_t| > 0$ at z_o.

It remains to discuss those portions of W on which $|u_z| = 0$. These are identical with the subsets of W on which $|\varphi_x + i\Psi_x| = 0$, and we shall use properties of the quasi-conformal mapping

(4.2.5) $$w = \varphi_x + i\Psi_x$$

to discuss their nature. Let us denote by Γ any region complementary to B+b in which

(4.2.6) $$0 < |\varphi_x + i\Psi_x| < \varepsilon ,$$

for some small $\varepsilon > 0$. The boundary $|\varphi_x + i\Psi_x| = \varepsilon$ of this region consists of a finite number of analytic arcs which transform by (2.5) into the circumference $|w| = \varepsilon$, covered finitely many times, say, m times. Since the mapping $w = \varphi_x + i\Psi_x$ is univalent in the small, we conclude that the image of Γ in the circle $0 < |w| < \varepsilon$ is a Riemann surface of m sheets. Hence the boundary of Γ contains at most m continua on which $|\varphi_x + i\Psi_x| = 0$, and we shall prove that these continua reduce to isolated points.

We remark that the quasi-conformal mapping (2.5) has a dilation quotient in Γ which is smaller than

(4.2.7) $$K = \max\left(\max y, \max \frac{1}{y}\right)$$

for all $z = x + iy$ in Γ. If now one of the continua $|\varphi_x + i\Psi_x| = 0$ bounding Γ were non-degenerate, there would exist a positive lower bound δ for the Dirichlet integral

$$\iint_{\Gamma} \left\{ U_x^2 + U_y^2 \right\} dxdy$$

of any function U in Γ which vanishes on the boundary portions $|\varphi_x + i\psi_x| = 0$ and is identically 1 on the boundary curves $|\varphi_x + i\psi_x| = \varepsilon$. We define such a function U by the formula

$$(4.2.8) \qquad U = \begin{cases} 0 & , \quad |w| \leq r \\ \log \frac{|w|}{r} \Big/ \log \frac{\varepsilon}{r} & , \quad |w| \geq r \end{cases} ,$$

where $0 < r < \varepsilon$. We can estimate its Dirichlet integral in terms of the quasi-conformal mapping $w = \varphi_x + i\psi_x$ to obtain (with $w = u + iv$)

$$(4.2.9) \qquad \begin{aligned} \delta &\leq \iint_{\Gamma} \left\{ U_x^2 + U_y^2 \right\} dxdy \\ &\leq mK \iint_{r < |w| < \varepsilon} \left\{ U_u^2 + U_v^2 \right\} dudv \\ &= 2\pi\, mK \int_r^{\varepsilon} \frac{1}{(\log \frac{\varepsilon}{r})^2} \frac{1}{\rho^2} \rho\, d\rho \\ &= \frac{2\pi\, mK}{(\log \frac{\varepsilon}{r})} . \end{aligned}$$

The estimate on the right approaches zero as $r \to 0$, and this gives a contradiction of the hypothesis that the continua $|\varphi_x + i\psi_x| = 0$ on the boundary of Γ do not degenerate to points.

In order to see quite clearly that the continua $|\varphi_x + i\psi_x| = 0$ of the set W all reduce to isolated points, we note that the analytic arcs of W in the region $|\varphi_x + i\psi_x| > 0$ can be given a certain order as we proceed from b to B along W. This follows from the fact that deleting a small arc from W must disconnect W, since it diminishes the virtual mass α. The two disconnected parts of W are composed of those arcs which come before and those which come after the deleted one, respectively. Thus any continuum of W on which $|\varphi_x + i\psi_x| = 0$ must be preceded and must be followed by arcs of W in the region

$|\varphi_x + i\psi_x| > 0$, and therefore must be a boundary component of some region Γ of the type (2.6). Hence any such continuum reduces to a point, and we can conclude that these points are removable singularities of the solution u of (1.9) defined by (2.2). For topologically more complicated problems of the type (1.4) than we have posed here, the function u can have a certain number of branch-points where u_z vanishes, but in the present case, because of the monotonic dependence of α on D, this possibility is excluded. The extremal continuum W must reduce to a simple arc without forks, since otherwise deletion of a branch of W, decreasing α, would be feasable. The arc W is an analytic curve, since it consists merely of a level curve $u = 0$ of a regular solution of (1.9).

It follows from the above arguments that the surface of revolution about the x -axis generated by W is analytic and that the speed $|\nabla\varphi|$ of the flow φ past this surface is continuous through it, so that the surface represents a vortex sheet in the flow. One checks that on either side of the vortex sheet, in the present example, the velocities have opposite directions. More complicated vortex sheets with several branches and forks can be constructed by the same method with relatively little additonal difficulty.

3. Uniqueness.

We sketch in this section a proof of the uniqueness of the extremal curve W solving (1.4) which is based on the ideas of Chapter III. A possibly shorter approach to the whole problem might be achieved through the Dirichlet principle, but we prefer here to emphasize the use of variational methods.

Let W^* be any curve joining b to the set B or the x -axis. The curves W and W^*, together with B and the x -axis, bound certain subregions D_ν of D. In each region D_ν, we let U be the harmonic function which vanishes on the

arcs of W bounding D_ν, which has the value 1 on the arcs of W^* bounding D_ν, and which has a vanishing normal derivative at any points of B or the x-axis bounding D_ν. We denote by W_t the aggregate of all level curves $U = t$ in the various regions D_ν and we denote by α_t the virtual mass of the flow (1.3) past the object $B+W_t+b$. In particular, $W_0 = W$ and $W_1 = W^*$, while $\alpha_0 = \alpha$, and $\alpha_1 = \alpha^*$ is the virtual mass coefficient corresponding to W^*.

From the formula

$$(4.3.1) \qquad \delta^2\alpha = \frac{1}{2}\int\left(\frac{\partial u}{\partial\nu}\right)^2 \frac{(\delta\nu)^2 ds}{\rho} + \iiint\left[\left\{\nabla(\delta u)\right\}^2 + \frac{3}{4y^2}(\delta u)^2\right]d\tau$$

for the second variation of α under a normal shift $\delta\nu$ of the boundary, we derive the expression

$$(4.3.2) \qquad \frac{d^2\alpha_t}{dt^2} = \iint_D\left[\left(\nabla\frac{du_t}{dt}\right)^2 + \frac{3}{4y^2}\left(\frac{du_t}{dt}\right)^2\right]d\tau$$

for the second derivative of α_t. By the continuity of $(\nabla u)^2$ across $W_0 = W$ and by (1.6), we have

$$(4.3.3) \qquad \left.\frac{d\alpha_t}{dt}\right|_{t=0} = 0 \quad ,$$

and by (3.2)

$$(4.3.4) \qquad \frac{d^2\alpha_t}{dt^2} > 0 \quad , \quad 0 \leq t \leq 1 \quad .$$

It follows that $\alpha_t > \alpha$ for $t > 0$, and, in particular,

$$(4.3.5) \qquad \alpha^* > \alpha \quad ,$$

provided W^* does not coincide with W. Formula (4.3.5) establishes that the extremal curve W for (1.4) is unique. A closer examination of the proof even shows that an arbitrary axially symmetric vortex sheet joining b to B has a meridian curve satisfying (1.4), and hence a vortex sheet such as we consider here must be unique.

CHAPTER V

VARIATION OF EIGENVALUES

1. Analytic dependence on parameters.

We consider now the eigenvalue problem of determining a non-trivial solution of the partial differential equation

$$\nabla^2 u + \lambda u = 0 \tag{5.1.1}$$

which vanishes on the boundary C of the plane domain D. We consider the eigenvalues λ as functionals of the varying domain D.

Hilbert [11] proved the continuous dependence of the eigenvalues λ on the coefficients of a differential equation if the basic domain D is kept fixed. We can utilize his method and his result by the artifice used in Chapter II. We transform the domain D into a domain D^* by means of a deformation (2.2.2) and ask for the eigenfunctions $u^*(x_1,x_2)$ and eigenvalues λ^* of the same equation (1.1) with respect to the new domain D^*. Then, we introduce the functions

$$U(x_1,x_2;\varepsilon) = u^*(x_1^*,x_2^*) \tag{5.1.2}$$

defined in the fixed domain D and treat the eigenvalue problem for this fixed domain arising from the transformation. We may describe $U(x_1,x_2;\varepsilon)$ as an eigenfunction in D with respect to the differential equation (see (2.2.9) and take $p = \rho = 0$)

$$L_\varepsilon[U] + \lambda^* \theta U = 0 \tag{5.1.3}$$

and the boundary condition $U = 0$ on C. In this way, the domain dependence of the eigenvalue λ^* is translated into its dependence on the coefficients of the new differential operator L_ε.

We may now proceed as in Chapter II and derive the variational formulas for the eigenvalues λ by a proper use of Green's identity. We may assert in view of (2.2.8) that all coefficients of the differential expression $L_\varepsilon[U]$ depend analytically upon the parameter ε. It is then easy to see that all non-degenerate eigenvalues λ^* and their corresponding eigenfunctions U depend analytically on ε, provided U is normalized.

In order to prove this statement we observe at first that the Green's function $g(P,Q;\varepsilon)$ belonging to the differential operator L_ε depends analytically on ε. In fact, the parametrix function $s(P,Q;\varepsilon)$ of this differential operator, defined for two independent variables in analogy with (2.3.5), may be chosen to depend analytically on ε for $P \neq Q$. The Green's function $g(P,Q;\varepsilon)$ may then be obtained in form of a Neumann's series which solves the integral equation (2.4.20) with the analytic kernel (2.4.21) in ε. Since the Neumann's series converges uniformly, we see that $g(P,Q;\varepsilon)$ depends analytically upon this parameter also, at least in a neighborhood of the value $\varepsilon = 0$.

The eigenfunctions $U(P;\varepsilon)$ of the differential equation (1.3) may also be considered as eigenfunctions of the integral equation

$$(5.1.4) \qquad U(P;\varepsilon) - \lambda^* \iint_D g(P,Q;\varepsilon) U(Q;\varepsilon) \theta(Q) d\tau_Q = 0$$

with the same eigenvalues λ^*. The eigenvalues λ^* appear as the roots of an entire function $D\{\lambda;\varepsilon\}$ in λ which is also analytic for sufficiently small ε. If λ^* is non-degenerate, we have $\frac{\partial}{\partial\lambda} D\{\lambda^*;\varepsilon\} \neq 0$ near the value $\varepsilon = 0$ and $\lambda^* = \ell(\varepsilon)$ is an analytic function of ε. Since the eigenfunctions $U(P;\varepsilon)$ can be represented as Fredholm minors which depend analytically on ε, we have thus also proved the analytic dependence of $U(P;\varepsilon)$ on ε, if the corresponding eigenvalue is non-degenerate.

We will apply this result in particular to the case of the lowest eigenvalue of the partial differential equations (1.3) and (1.1) and its corresponding eigenfunction. It is well known that the lowest eigenvalue of an equation (1.1) is non-degenerate and consequently we may develop $U(P;\varepsilon)$ and $\lambda^*(\varepsilon)$ into power series in ε which converge for small values of ε. Henceforth, $\lambda^*(\varepsilon)$ and $U(P;\varepsilon)$ shall always denote the lowest eigenvalue and its corresponding eigenfunction. It is obvious how to generalize the following considerations to the case of a non-degenerate eigenvalue of arbitrary order.

Since we can put

$$\lambda^*(\varepsilon) = \lambda_o + \varepsilon\lambda_1 + \frac{1}{2}\varepsilon^2\lambda_2 + \dots \quad , \tag{5.1.5}$$

$$U(P;\varepsilon) = u_o(P) + \varepsilon u_1(P) + \frac{1}{2}\varepsilon^2 u_2(P) + \dots \quad , \tag{5.1.6}$$

where λ_o and $u_o(P)$ are the eigenvalue and eigenfunction of the original domain D, we may calculate all other terms λ_ν and $u_\nu(P)$ of the above series development by inserting these series into (1.3) and comparing the coefficients of equal powers of ε. This procedure is called in physics the "perturbation method" and is widely used in applications. It is easy to handle and satisfactory in most problems in applied mathematics. Obviously, it works only in the case of analytic coefficients in the differential equation. We can now apply this procedure in order to study the dependence of the eigenvalues upon their domain of definition.

We introduce for this purpose the tensor

$$U_{ik}(P) = 2\,\frac{\partial u_o}{\partial x_i}\,\frac{\partial u_o}{\partial x_k} - \delta_{ik}(\nabla u_o)^2 \quad , \tag{5.1.7}$$

which is symmetric and satisfies the equations

$$\sum_{k=1}^{2}\frac{\partial U_{ik}}{\partial x_k} = -\lambda_o\,\frac{\partial}{\partial x_i}\,u_o^2 \quad . \tag{5.1.8}$$

We obtain the following formula similar to (2.6.6):

$$(5.1.9) \qquad \left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0} = -\iint_D \left\{ \sum_{i,k=1}^{2} U_{ik} \frac{\partial S_i}{\partial x_k} + \sum_{k=1}^{2} \frac{\partial}{\partial x_k} (\lambda_o S_k) \cdot u_o^2 \right\} d\tau \quad .$$

It is easily seen that by virtue of the equations (1.8) the integrand is a divergence term and that the expression (1.9) may be reduced to an integral extended over the boundary C of D. In this way, we obtain in analogy with (2.6.15)

$$(5.1.10) \qquad \delta\lambda = \int_C \left(\frac{\partial u}{\partial \nu}\right)^2 \delta\nu \, ds \quad .$$

This formula is due to Rayleigh and is derived here from the method of interior variations. If we use the complex notation (2.8.6) introduced in Section 8 of Chapter II, we find

$$(5.1.11) \qquad \left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0} = -\operatorname{Re}\left\{ \iint_D [8 \frac{\partial F}{\partial \bar{z}} \left(\frac{\partial u_o}{\partial z}\right)^2 + 2 \frac{\partial}{\partial z} (\lambda_o F) u_o^2] d\tau \right\} \quad .$$

Let us remark finally that the formula (1.10) exhibits clearly the monotonic dependence of the eigenvalue λ upon the domain D. We see that for $\delta\nu > 0$, that is, for a shrinking domain D, the eigenvalue λ increases. Analogous formulas can be given for all higher non-degenerate eigenvalues of the differential equation (3.1) and similar conclusions can be drawn.

2. The Hilbert-Green function.

In this section we will calculate the first coefficient $u_1(P)$ in the development (1.6) of the eigenfunction $U(P;\varepsilon)$. In this way, we will also prepare the determination of the term λ_2 which will give the formula for the second variation of the eigenvalue.

In order to calculate u_1, we define the Hilbert-Green function $\Gamma(P,Q)$ for the differential equation (1.1) and the domain D as follows:

a) The function $\Gamma(P,Q)$ is twice continuously differentiable in D, except at $P = Q$. It satisfies the inhomogeneous partial differential equation

$$(5.2.1) \qquad L_o[\Gamma(P,Q)] + \lambda_o \Gamma(P,Q) = \nabla_P^2 \Gamma(P,Q) + \lambda_o \Gamma(P,Q)$$
$$= u_o(P) u_o(Q) \qquad .$$

b) The function $\Gamma(P,Q) + \frac{1}{2\pi} \log \overline{PQ}$ is continuously differentiable in D.

c) $\Gamma(P,Q)$ vanishes for $P \in C$, $Q \in D$.

The Hilbert-Green function has to be applied to differential operators for which a proper Green's function does not exist, as is the case for (1.1). The existence of this function can easily be shown from the basic theorems on integral equations. In fact, let $G(P,Q)$ be the Green's function of D with respect to the differential equation $L_o[u] = 0$. We have obviously

$$(5.2.2) \qquad u_o(P) = \lambda_o \iint_D G(P,Q) u_o(Q) d\tau_Q \qquad ,$$

and $\Gamma(P,Q) - G(P,Q) = H(P,Q)$ is a continuously differentiable function in D which vanishes for $P \in C$. $H(P,Q)$ satisfies the differential equation

$$(5.2.3) \qquad \nabla_P^2 H(P,Q) + \lambda_o H(P,Q) = u_o(P) u_o(Q) - \lambda_o G(P,Q) \qquad .$$

If we can show conversely that (2.3) has a continuous solution H in D which vanishes for $P \in C$, we can construct $\Gamma(P,Q) = G(P,Q) + H(P,Q)$. The necessary and sufficient condition for the existence of a solution of the inhomogeneous differential equation (2.3) which vanishes on the boundary C of D is the orthogonality of its right-hand side to the eigenfunction $u_o(P)$. But this condition is fulfilled in view of (2.2) and the existence of the Hilbert-Green function has thus been established.

We observe that the above conditions (a), (b), (c) do not yet determine $\Gamma(P,Q)$ uniquely, since the addition of any multiple of $u_o(P)$ to $\Gamma(P,Q)$ does not affect these requirements. We determine $\Gamma(P,Q)$ uniquely by the additional condition

$$(5.2.4) \qquad \iint_D \Gamma(P,Q)u_o(Q)d\tau_Q = 0 \quad .$$

It can be shown in the usual way that the Hilbert-Green function $\Gamma(P,Q)$, which is now unique, is symmetric in both its argument points.

Consider now a function $v(P)$ which satisfies the inhomogeneous equation

$$(5.2.5) \qquad L_o[v] + \lambda_o v(P) = f(P) \quad , \quad v(P) = 0 \quad \text{for} \quad P \in C \quad .$$

We have by Green's identity and the vanishing of Γ and v on C

$$(5.2.6) \qquad \iint_C \left\{ vL_o[\Gamma] - \Gamma L_o[v] \right\} d\tau_Q = v(P) \quad .$$

Using now (2.1) and (2.5), we obtain the following representation for $v(P)$:

$$(5.2.7) \qquad v(P) = -\iint_D \Gamma(P,Q)f(Q)d\tau_Q + u_o(P)\iint_D u_o(Q)v(Q)d\tau_Q \quad .$$

This result is analogous to the solution (2.1.6) of the corresponding inhomogeneous equation (2.1.5) in the case when a proper Green's function exists. We observe that equation (2.5) determines $v(P)$ only up to a multiple of the eigenfunction $u_o(P)$, which is clearly shown by the solution formula (2.7). We remark also that $f(P)$ cannot be prescribed arbitrarily, but must be orthogonal to the eigenfunction $u_o(P)$.

For later applications we want to prove that the quadratic functional

$$(5.2.8) \qquad \mathcal{L}\left\{f,f\right\} = \iint_D \iint_D \Gamma(P,Q)f(P)f(Q)d\tau_P d\tau_Q$$

based on the Hilbert-Green function is positive semi-definite and that it vanishes only if $f(P)$ is a multiple of the eigenfunction $u_o(P)$. We may

put every function f(P) into the form

$$(5.2.9) \qquad f(P) = u_o(P)\iint_D f u_o d\tau + f_1(P) \quad ,$$

where $f_1(P)$ is orthogonal to $u_o(P)$. Because of (2.4), we have

$$(5.2.10) \qquad \mathcal{L}\{f,f\} = \mathcal{L}\{f_1,f_1\} \quad ,$$

so that our assertion will be proved if we show that $\mathcal{L}\{f,f\}$ is positive-definite for all functions f(P) which are orthogonal to $u_o(P)$.

For all such functions f(P) the inhomogeneous equation (2.5) has solutions v(P) of the form (2.7). Hence, we may write

$$(5.2.11) \qquad \mathcal{L}\{f,f\} = -\iint_D f(P)v(P)d\tau_P = -\iint_D (L_o[v] + \lambda_o v)v d\tau \quad .$$

Applying finally Green's identity, we obtain

$$(5.2.12) \qquad \mathcal{L}\{f,f\} = \iint_D [(\nabla v)^2 - \lambda_o v^2] d\tau \quad .$$

It is well known that we can characterize the first eigenvalue of the differential equation (1.1) as the minimum value of the ratio

$$\iint_D [\nabla v^2] d\tau \Big/ \iint_D v^2 d\tau$$

for all continuously differentiable functions v(P) in D which vanish on the boundary C. The minimum value is accepted only if v(P) is a multiple of the eigenfunction $u_o(P)$. Hence, we conclude from (2.12) that

$$(5.2.13) \qquad \mathcal{L}\{f,f\} \geqslant 0$$

for all functions f(P) which are orthogonal to $u_o(P)$. Equality can hold in (2.13) only if v(P) is a multiple of $u_o(P)$, which implies by (2.5) that $f(P) \equiv 0$. This shows the positive-definite character of $\mathcal{L}\{f,f\}$.

We apply now the solution formula (2.7) to the equation

$$(5.2.14) \qquad L_o[u_1] + \lambda_o u_1 = -\lambda_1 u_o - L_o'[u_o] - \lambda_o u_o\left(\frac{\partial S_1}{\partial x_1} + \frac{\partial S_2}{\partial x_2}\right)$$

obtained by comparing ε terms in the relation found when we insert (1.5) and (1.6) into (1.3). This gives

$$(5.2.15) \quad u_1(P) = \iint_D [L_o'[u_o(Q)] + \lambda_o u_o(Q)\left(\frac{\partial S_1}{\partial \xi_1} + \frac{\partial S_2}{\partial \xi_2}\right)]\,\Gamma(P,Q)\,d\tau_Q + \mu u_o(P) \quad ,$$

where

$$(5.2.16) \qquad \mu = -\frac{1}{2}\iint_D u_o^2 \left(\frac{\partial S_1}{\partial x_1} + \frac{\partial S_2}{\partial x_2}\right) d\tau \quad .$$

We can simplify the result (2.15) if we introduce the symmetric variational tensor

$$(5.2.17) \qquad V_{ik}(P;Q) = \frac{\partial u_o(P)}{\partial x_i}\,\frac{\partial \Gamma(P,Q)}{\partial x_k} + \frac{\partial u_o(P)}{\partial x_k}\,\frac{\partial \Gamma(P,Q)}{\partial x_i}$$

$$- \delta_{ik} \nabla u_o(P) \cdot \nabla \Gamma(P,Q) \quad .$$

We may then bring (2.15) into the form

$$(5.2.18) \quad u_1(Q) = \iint_D \left\{ \sum_{i,k=1}^{2} V_{ik}(P;Q)\,\frac{\partial S_i}{\partial x_k} + \sum_{k=1}^{2} \frac{\partial}{\partial x_k}(\lambda_o S_k) u_o \Gamma(P;Q) \right\} d\tau_P$$

$$+ \mu u_o(Q) \quad .$$

We have the differential equations

$$(5.2.19) \quad \sum_{k=1}^{2} \frac{\partial}{\partial x_k} V_{ik}(P;Q) = -\lambda_o \frac{\partial}{\partial x_i}[u_o(P)\,\Gamma(P;Q)] + \frac{1}{2} u_o(Q) \frac{\partial}{\partial x_i} u_o(P)^2 \;.$$

Hence, we may bring (2.18) into the divergence form

$$(5.2.20) \quad u_1(Q) = \iint_D \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{2} V_{ik}(P;Q) S_i + \lambda_o S_k u_o(P)\,\Gamma(P;Q) \right\} d\tau_P$$

$$- \frac{1}{2} u_o(Q) \iint_D \sum_{k=1}^{2} \frac{\partial}{\partial x_k}(S_k u_o^2)\,d\tau \quad .$$

We can reduce this domain integral to a line integral extended over the boundary curve C of D. We use the fact that Γ and u_o vanish on C and take into account the singularity of the tensor V_{ik} at the point Q. We obtain

$$(5.2.21)\qquad u_1(Q) = \sum_{i=1}^{2} s_i(Q)\,\frac{\partial u_o(Q)}{\partial \xi_i} - \int_C \frac{\partial \Gamma(P,Q)}{\partial \nu_P}\,\frac{\partial u_o(P)}{\partial \nu_P}\,(S\cdot\nu)ds_P \quad .$$

Let us return to the eigenfunction $u^*(Q;\varepsilon)$ of the original differential equation (1.1) with respect to the variable domain D^*. Since

$$(5.2.22)\qquad U(x_1,x_2;\varepsilon) = u^*(x_1+\varepsilon S_1,x_2+\varepsilon S_2;\varepsilon) \quad ,$$

we derive from (2.21)

$$(5.2.23)\qquad \left.\frac{du^*(Q;\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = -\int_C \frac{\partial \Gamma(P,Q)}{\partial \nu_P}\,\frac{\partial u_o(P)}{\partial \nu_P}\,(S\cdot\nu)ds_P \quad ,$$

or, in equivalent variational form

$$(5.2.24)\qquad \delta u(Q) = -\int_C \frac{\partial \Gamma(P,Q)}{\partial \nu_P}\,\frac{\partial u}{\partial \nu_P}\,\delta\nu\, ds \quad .$$

The function

$$(5.2.25)\qquad h(Q) = -\int_C \frac{\partial \Gamma(P,Q)}{\partial \nu_P}\,\frac{\partial u_o(P)}{\partial \nu_P}\,(S\cdot\nu)ds_P$$

will play a role in the second variation of the eigenvalue λ. We observe that because of (1.10) and (2.1)

$$(5.2.26)\qquad L_o[h(Q)] + \lambda_o h(Q) = -\lambda_1 u_o(Q) \quad ,$$

and we verify easily that h(Q) has on C the boundary values

$$(5.2.27)\qquad h(Q) = -\frac{\partial u_o}{\partial \nu}\,(S\cdot\nu) = -\sum_{i=1}^{2} s_i(Q)\,\frac{\partial u_o(Q)}{\partial \xi_i} \quad .$$

3. Second variation of the eigenvalue.

We are now able to calculate the coefficient λ_2 in the series development (1.5) for the eigenvalue $\lambda^*(\varepsilon)$ under a deformation (2.2.2) corresponding to a normal shift of C. It is possible to achieve by a laborious process of rearrangement of terms and of integration by parts a rigorous derivation very similar to those performed in Chapter II. The end result of the calculations can be obtained easily, however, by the following heuristic argument.

We suppose that C is an analytic curve and that $u(P)$ and $u^*(P)$ are both regular functions in the original domain D and satisfy the differential equations

(5.3.1) $$\Delta^2 u + \lambda u = 0 \quad , \quad \Delta^2 u^* + \lambda^* u^* = 0$$

in D. We apply Green's identity

(5.3.2) $$\iint_D \left\{ u^*[\Delta^2 u + \lambda u] - u[\Delta^2 u^* + \lambda^* u^*] \right\} d\tau = \int_C \left(u \frac{\partial u^*}{\partial \nu} - u^* \frac{\partial u}{\partial \nu}\right) ds + (\lambda - \lambda^*) \iint_D u u^* d\tau \quad .$$

Since $u(P)$ vanishes on C and in view of (3.1), we obtain

(5.3.3) $$(\lambda^* - \lambda) \iint_D u u^* d\tau = \int_C u^* \frac{\partial u}{\partial \nu} ds \quad .$$

We observe that because of (2.23) and (2.25)

(5.3.4) $$u^*(P) = u(P) + \varepsilon h(P) + o(\varepsilon) \quad .$$

Assuming $u^*(P)$ vanishes on a deformed curve C^*, we have

(5.3.5) $$0 = u^*(P^*) = u^*(P) + \frac{\partial u^*(P)}{\partial \nu} \varepsilon N(s) + \frac{1}{2} \frac{\partial^2 u^*(P)}{\partial \nu^2} \varepsilon^2 N(s)^2 + \dots ,$$

where $\varepsilon N(s)$ represents the amount of the normal shift from C to C^* at the point P. Hence

$$(5.3.6)\quad (\lambda^{*}-\lambda)[1+\varepsilon\iint_D hud\tau+o(\varepsilon)]$$

$$= -\int_C \frac{\partial u^{*}}{\partial\nu}\frac{\partial u}{\partial\nu}\,\varepsilon N ds - \frac{1}{2}\int_C \frac{\partial^2 u}{\partial\nu^2}\frac{\partial u}{\partial\nu}\,\varepsilon^2 N^2 ds + o(\varepsilon^2)$$

$$= -\int_C \left(\frac{\partial u}{\partial\nu}\right)^2 \varepsilon N ds - \frac{1}{2}\int_C \frac{\partial^2 u}{\partial\nu^2}\frac{\partial u}{\partial\nu}\,\varepsilon^2 N^2 ds - \varepsilon^2\int_C \frac{\partial h}{\partial\nu}\frac{\partial u}{\partial\nu} N ds$$

$$+ o(\varepsilon^2)\quad .$$

We observe that the last right-hand integral can be written in view of (2.27) and (2.26) as

$$(5.3.7)\quad \varepsilon^2\int_C \frac{\partial h}{\partial\nu} h ds = -\varepsilon^2\iint_D[(\nabla h)^2-\lambda_o h^2]d\tau + \lambda_1\iint_D hud\tau\quad .$$

Finally, we put $\lambda^{*}=\lambda+\varepsilon\lambda_1+\frac{\varepsilon^2}{2}\lambda_2+\ldots$ and find by use of (1.10) and by comparing the coefficients of ε^2 on both sides of (3.6)

$$(5.3.8)\quad \lambda_2 = -\int_C \frac{\partial^2 u}{\partial\nu^2}\frac{\partial u}{\partial\nu} N(s)^2 ds - 2\iint_D[(\nabla h)^2-\lambda_o h^2]d\tau\quad .$$

Since u(P) satisfies the first equation (3.1) and vanishes on C, we have

$$(5.3.9)\qquad \frac{\partial^2 u}{\partial\nu^2} = \frac{1}{\rho}\frac{\partial u}{\partial\nu}\quad ,$$

where $\rho(s)$ is the radius of curvature at the point P with the parameter value s. Hence, we may bring (3.8) also into the form

$$(5.3.10)\quad \lambda_2 = -\int_C \left(\frac{\partial u}{\partial\nu}\right)^2 \frac{N(s)^2}{\rho} ds - 2\iint_D[(\nabla h)^2-\lambda_o h^2]d\tau\quad .$$

In general, h(P) will not vanish identically on C and we cannot assert that the second right-hand integral is non-negative. Thus, even in the case of a convex curve C ($\rho \geqslant 0$) we cannot be sure that λ_2 is negative.

We shall consider in the next sections variations of a more special form for which the convex dependence of λ upon the variation parameter can be shown. It will appear that interior variational formulas are more convenient for this purpose than formula (3.10).

4. Convexity.

The variational formulas for the eigenvalue λ and the eigenfunction $u(P)$ become very simple if we assume that the complex vector

$$F(z,\bar{z}) = S_1(x_1,x_2) + iS_2(x_1,x_2)$$

is an analytic function of the complex variable z. In this case, we have by (1.11)

$$(5.4.1) \qquad \left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0} = -2\lambda_o \operatorname{Re}\left\{\iint_D F'(z)u_o(z)^2 d\tau\right\} .$$

The analogous formula for $d^2\lambda/d\varepsilon^2$ at $\varepsilon=0$ under a variation $z^* = z+\varepsilon F(z)$ is more difficult to derive from (3.10), because such a deformation is not equivalent to a normal shift when second order terms are involved. But for a conformal mapping, $L_\varepsilon[U] \equiv \nabla^2 U$ and $\Theta = |1+\varepsilon F'|^2$, so that substitution of (1.5) and (1.6) into (1.3) and comparison of terms in ε^2 yields

$$(5.4.2) \quad \nabla^2 u_2 + \lambda_o u_2 = -2\lambda_1 u_1 - 4\lambda_o u_1 \operatorname{Re}\{F'\} - \lambda_2 u_o - 4\lambda_1 u_o \operatorname{Re}\{F'\} - 2\lambda_o u_o |F'|^2 .$$

Since u_2 vanishes on C, the eigenfunction u_o must be orthogonal to the right-hand side of (4.2), and this gives

$$(5.4.3) \quad \lambda_2 = -4\lambda_1 \iint_D u_o^2 \operatorname{Re}\{F'\}\, d\tau - 2\lambda_o \iint_D u_o^2 |F'|^2 d\tau - 2\lambda_1 \iint_D u_o u_1 d\tau$$

$$- 4\lambda_o \iint u_o u_1 \operatorname{Re}\{F'\}\, d\tau .$$

Using the condition

$$(5.4.4) \qquad \nabla^2 u_1 + \lambda_o u_1 = -\lambda_1 u_o - 2\lambda_o u_o \operatorname{Re}\{F'\}$$

for ε terms in (1.3) to cancel and noting that with

$$(5.4.5) \qquad f = \lambda_1 u_o + 2\lambda_o u_o \operatorname{Re}\{F'\}$$

formula (2.12) gives

$$(5.4.6) \qquad \mathcal{L}\{f,f\} = \iint_D [(\nabla u_1)^2 - \lambda_o u_1^2] d\tau ,$$

we bring (5.4.3) into the form

$$\lambda_2 = -2\mathcal{L}\{f,f\} - 2\lambda_o \iint_D u_o^2 |F'|^2 d\tau - 4\lambda_1 \iint_D u_o^2 \operatorname{Re}\{F'\} d\tau \quad . \tag{5.4.7}$$

Using (4.1), we simplify (4.7) to obtain

$$\lambda \frac{d^2\lambda}{d\varepsilon^2} - 2\left(\frac{d\lambda}{d\varepsilon}\right)^2 \Big|_{\varepsilon=0} = -2\lambda_o \mathcal{L}\{f,f\} - 2\lambda_o^2 \iint_D |F'|^2 u_o^2 d\tau \quad . \tag{5.4.8}$$

We observe that the right-hand side of (4.8) is always negative, except for the case $F'(z) \equiv 0$; the deformation induced in this exceptional case is a translation of the whole plane, and it is obvious that in this case, indeed, $\lambda(\varepsilon) = \lambda(0)$. Thus, we are led to the differential inequality

$$\frac{d^2}{d\varepsilon^2}\left[\frac{1}{\lambda(\varepsilon)}\right] = \frac{-1}{\lambda^3}\left[\frac{d^2\lambda}{d\varepsilon^2} - 2\left(\frac{d\lambda}{d\varepsilon}\right)^2\right] \geqslant 0 \quad , \tag{5.4.9}$$

which proves the convexity of the reciprocal of the eigenvalue under infinitesimal analytic deformations.

In order to generalize this result to finite deformations, we consider the partial differential equation

$$\nabla^2 u + \lambda |h'(z,T)|^2 u = 0 \quad , \quad u = 0 \ \text{ on } \ C \quad , \tag{5.4.10}$$

where $h(z,T)$ is an analytic function of the complex variable $z \in D$ and the real parameter T, $0 \leq T \leq 1$. We set $h' = \frac{\partial h}{\partial z}$ and assume $h'(z,0) = 1$. Obviously, (4.10) is the membrane equation for a domain D_T obtained from D by a mapping $h(z,T)$, referred back to the domain D. We observe that the differential equation can be treated independently of the univalency of $h(z,T)$ and permits a definition of eigenfunctions and eigenvalues of the membrane problem also for non-schlicht domains. We normalize the eigenfunctions $u(z;T)$ of (4.10) by the condition

$$\iint_D u^2 |h'(z,T)|^2 d\tau = 1 \tag{5.4.11}$$

and consider $u_T = u(z;T)$ and $\lambda_T = \lambda(T)$ in their dependence on T. We may repeat all calculations of this chapter leading to expressions for $\frac{d\lambda}{dT}$ and $\frac{d^2\lambda}{dT^2}$, but now these derivatives can be determined for finite values of T.

We introduce a Hilbert-Green function $\Gamma_T(z,\zeta)$ by the conditions

$$\nabla^2 \Gamma_T(z,\zeta) + \lambda_T |h'(z,T)|^2 \Gamma_T(z,\zeta) = |h'(z,T)|^2 u_T(z) u_T(\zeta) \quad , \tag{5.4.12}$$

$$\Gamma_T(z,\zeta) + \frac{1}{2\pi} \log |z-\zeta| \tag{5.4.13}$$

is continuously differentiable in D,

$$\Gamma_T(z,\zeta) = 0 \quad \text{for} \quad z \in C, \ \zeta \in D \quad , \tag{5.4.14}$$

$$\iint_D \Gamma_T(z,\zeta) u_T(z) |h'(z,T)|^2 d\tau = 0 \quad . \tag{5.4.15}$$

The existence of such a fundamental function can be shown in exactly the same manner as was done in Section 3. Let $\mathcal{L}_T\{f,f\}$ be the quadratic functional defined in (2.8), but with Γ_T instead of Γ as kernel. We see immediately that $\mathcal{L}_T\{f,f\}$ is also semi-definite and vanishes only for the eigenfunction u_T.

One finds by easy computation

$$\frac{d\lambda}{dT} = -\lambda \iint_D u_T(z)^2 \frac{\partial}{\partial T} (|h'(z,T)|^2) d\tau \tag{5.4.16}$$

and

$$\frac{d^2\lambda}{dT^2} - 2 \frac{1}{\lambda(T)} \left(\frac{d\lambda}{dT}\right)^2 + 2\mathcal{L}_T\{f_T,f_T\} + \lambda(T) \iint_D \frac{\partial^2}{\partial T^2} \left\{|h'(z,T)|^2\right\} u_T(z)^2 d\tau = 0 \quad , \tag{5.4.17}$$

with

$$f_T = u_T(z)\left[\frac{d\lambda}{dT} |h'(z,T)|^2 + \lambda(T) \frac{\partial}{\partial T} (|h'(z,T)|^2)\right] \quad . \tag{5.4.18}$$

Thus, we may write

$$\frac{d^2}{dT^2} \left(\frac{1}{\lambda(T)}\right) = \frac{2}{\lambda^2} \mathcal{L}_T\{f_T,f_T\} + \frac{1}{\lambda} \iint_D \frac{\partial^2}{\partial T^2} |h'(z,T)|^2 u_T^2 \, d\tau \quad . \tag{5.4.19}$$

The convexity of $\lambda(T)$ is thus ensured if we can assert that

$$\frac{\partial^2}{\partial T^2} |h'(z,T)|^2 \geqslant 0, \quad \text{for} \quad z \in D \quad . \tag{5.4.20}$$

This is, for example, the case if

$$h(z,T) = z + TF(z) \quad , \tag{5.4.21}$$

since we have then

$$\frac{\partial^2}{\partial T^2} |h'(z,T)|^2 = 2|F'(z)|^2 \quad . \tag{5.4.22}$$

We have shown that the reciprocal $\lambda(T)^{-1}$ of the eigenvalue of a family of domains D_T is a convex function of the parameter T if the domains are obtained from the original domain D by the conformal mapping (4.21). It is not necessary that all domains D_T be schlicht in the complex plane.

5. Application to extremal problems.

As an application we consider the family $\mathcal{F}$ of univalent functions $f(z)$ in the unit circle with $f(0) = 0$, $f'(0) = 1$, and ask for a function $f_m(z) \in \mathcal{F}$ which leads to a domain Δ with maximum first eigenvalue λ of the membrane problem. We introduce the function $h(z,T) = z + T[f_m(z) - z]$ and the corresponding eigenvalue λ_T. We find from (4.16) and (4.19)

$$\frac{d}{dT}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda} \iint_D u_T^2 [2 \operatorname{Re}\left\{f_m'(z) - 1\right\} + 2T|f_m' - 1|^2] d\tau \quad , \tag{5.5.1}$$

$$\frac{d^2}{dT^2}\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda^2} \mathcal{L}_T + \frac{2}{\lambda} \iint_D u_T^2 \, |f_m' - 1|^2 d\tau \geqslant 0 \quad . \tag{5.5.2}$$

Observe that the first derivative of $\frac{1}{\lambda}$ vanishes for $T = 0$; this follows from the fact that $f_m'(z) - 1$ vanishes at the origin and from the radial symmetry of the first eigenfunction for the circle. Hence by (5.2), $\frac{d}{dT}(1/\lambda)$ will be positive for all $T > 0$ and we find

$$\frac{1}{\lambda(1)} \geqslant \frac{1}{\lambda_0} \quad , \tag{5.5.3}$$

if λ_o is the lowest eigenvalue of the unit circle. But since $\lambda(1)$ is the largest possible value of λ, we conclude that $f_m(z) \equiv z$; that is, among all equivalent domains the unit circle has the largest eigenfrequency. This result is due to Polya - Szegö [16].

We obtain the following corollary: If a domain D satisfies the condition

$$\iint_D u^2 F(z) d\tau = 0 \tag{5.5.4}$$

for all functions $F(z)$ regular in D and vanishing at a point $P \in D$, where u is the first eigenfunction of D, then D is necessarily a circle with P as center.

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